Reversible Coalescing-Fragmentating Wasserstein Dynamics on the Real Line

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Abstract We introduce a family of reversible fragmentating-coagulating processes of particles of varying size-scaled diffusivity with strictly local interaction on the real line as mathematically rigorous description of colloidal motion of fluids. The associated measure valued process provides a weak solution of a corrected Dean-Kawasaki equation for supercooled liquids without dissipation. Our construction is based on the introduction and analysis of a fundamentally new family of equilibrium measures for the associated dynamics and their Dirichlet forms. We identify the intrinsic metric as the quadratic Wasserstein distance, which makes the process a second non-trivial example of Wasserstein diffusion.

Keywords Wasserstein Diffusion · Varadhan Formula · Dean-Kawasaki-Equation

Mathematics Subject Classification (2000) 60J45 · 60H15 · 60J50 · 47D07 · 60J60

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1 Introduction and Statement of main results

1.1 Motivation

This paper is a continuation in a series of studies started in [46] when we asked for natural generalizations of Brownian motion of a single point to the case of an infinite or diffuse interacting particle system with conserved total mass. As critical consistency condition with respect to the trivial case of the empirical (Dirac) measure following a single Brownian motion we put the requirement that the local fluctuations of any such probability measure valued diffusion $\{\mu_t\}_{t\geq 0} \in \mathcal{P}(\mathbb{R}^d)$ be governed by a Varadhan formula of the form

$$\mathbb{P}\{\mu_{t+\varepsilon} \in A\} \sim \exp\left(-\frac{d^2_{\mathcal{W}}(\mu_t, A)}{2\varepsilon}\right), \quad \varepsilon \ll 1, \quad A \subset \mathcal{P}(\mathbb{R}^d),$$

where $d_{\mathcal{W}}$ denotes the quadratic Wasserstein distance on $\mathcal{P}(\mathbb{R}^d)$.

Physically, this means that the spatial fluctuations of such a measure valued process $\mu_\cdot$ should become high at locations where density of μ_t is low and vice versa, i.e. scaling of diffusivity is inverse proportional to density. On the level of mathematical heuristics we can combine the required Wasserstein Varadhan formula with Otto's formal infinite dimensional Riemannian picture of optimal transport [43] to obtain SPDE models of the form

$$d\mu_t = F(\mu_t)dt + \operatorname{div}(\sqrt{\mu}_t dW_t), \quad \mu_t \in \mathcal{P}(\mathbb{R}^d),$$

where $dW_\cdot$ is a white noise vector field on $\mathbb{R}^d$ and F is a model dependent drift operator. The canonical choice

$$F(\mu_t) = \beta \Delta \mu_t, \quad \beta \geq 0,$$

yields the so called *Dean-Kawasaki equation* for supercooled liquids appearing in the physics literature [14, 29, 30, 8, 15, 16, 48, 40, 54] (see also [12, 11, 22] for the regularised versions of the Dean-Kawasaki equation) but in [34, 33] we show that this equation is either trivial or ill posed, depending on the value of β. However, as shown in [46, 4], in $d = 1$ for $\beta > 0$, and more recently in [35] for $\beta = 0$, the model has non-trivial martingale solutions if one admits a certain additional nonlinear drift operator $\Gamma_\beta(\mu_t)dt$ as correction. The correction is the same for all $\beta > 0$ such that we arrive at the family of models

$$d\mu_t = \beta \Delta \mu_t dt + \Gamma_i(\mu_t)dt + \operatorname{div}(\sqrt{\mu}_t dW_t),$$

where $i \in \{0, 1\}$ depending whether $\beta = 0$ or $\beta > 0$. The two expressions for Γ_0 and Γ_1 are similar, but the constructions of the solutions for the two cases are very different. In [46] we use abstract Dirichlet form methods, in [35] we construct an explicit system of a continuum of coalescing Brownian particles of infinitesimal initial mass which slow (i.e. cool) down as they aggregate to bigger and bigger macro-particles before they eventually collapse to a single Brownian motion. At positive time the system consists of finitely many particles of different sizes almost surely, such that the distribution

$$\Gamma_0(\mu_t) = \frac{1}{2} \sum_{z \in \mathrm{supp}(\mu_t)} (\delta_z)''$$

is well defined for $t > 0$.

The point of departure of this work is the question whether there is a reversible counterpart to the coalescing particle model for the $\beta = 0$ case. In terms of the analogy to the Arratia flow [6] (see also [7, 21, 18, 19, 47, 51, 42, 53, 59, 56, 57, 36, 37, 23, 55, 49]) this means that we ask for a Brownian Net [55] type extension of the modified Arratia flow from [32, 41, 35] which should then include also particle break-ups but still satisfies the characteristic scaling requirement regarding the diffusivity of the aggregate particles. We note that very recently a particle model without interaction in dimension $d \geq 2$ which satisfies a similar martingale problem was considered in [50].

1.2 Heuristic Description of the Model

The main result of this work is an affirmative answer. We give it by constructing in rather explicit way a new family of measure valued processes on the real line which solve the same martingale problem for $\beta = 0$ and $\Gamma_i = \Gamma_0$ as the modified Arratia flow in [35], which satisfy the Wasserstein Varadhan formula and which are reversible. In this sense the new processes interpolate between the two previously known models.

As in the case of the modified Arratia flow, the model describes the motion of an uncountable collection of particles which are parametrized by the unit interval as index set and move on the real axis. It is assumed that the initial parametrization is monotone in particle location. The dynamics will preserve the monotone alignment, hence a state of the system at time t is given by a monotone real function $X_t : (0,1) \mapsto \mathbb{R}$, i.e. $X_t(u)$ is the position of particle u at time t. The corresponding empirical measure of the state is given by $\mu_t := (X_t)_\#(\mathrm{Leb}) \in \mathcal{P}(\mathbb{R})$ (image measure of Lebesgue measure Leb on $[0,1]$ under X_t). We call the atoms of μ_t empirical particles, the size of an atom located in $x \in \mathbb{R}$ at time t given by $m(x,t) = \mathrm{Leb}\{u \in (0,1) : \ X_t(u) = x\}$.

The basic idea for the construction of $\mu_\cdot$ is to use (sticky) reflection interaction when particles are at the same location. As for the 'stickiness', particles sitting at the same location will be subject to the same random, i.e. Gaussian perturbation of their location. Since they share a common perturbation the net volatility of this perturbation is scaled in inverse proportional way by the total mass of particles occupying the same spot, i.e. the size of the empirical particle at that location. Second, the random perturbations at different spots are independent.

For the 'reflection' part of the interaction we assign once and for all times to each particle a certain number

$$[0,1] \ni u \mapsto \xi(u) \in \mathbb{R},$$

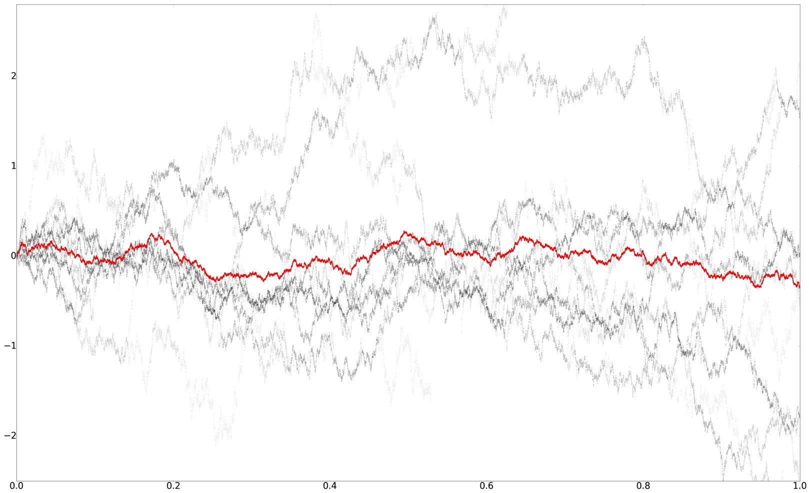
which we call its *interaction potential.* The function ξ is a free parameter of the model.

In addition to the random forcing described above, each particle will also experience a drift force given by the difference between its own interaction potential and the average interaction potential among all particles occupying the same location. As a consequence, if all occupants of a certain spot have the same interaction potential, none of them will feel any drift. (As they also share the same random forcing, in this case they will move but stay together for all future times.) Conversely, big differences in interaction potential lead to strong drift apart among the particles sitting at the same location.

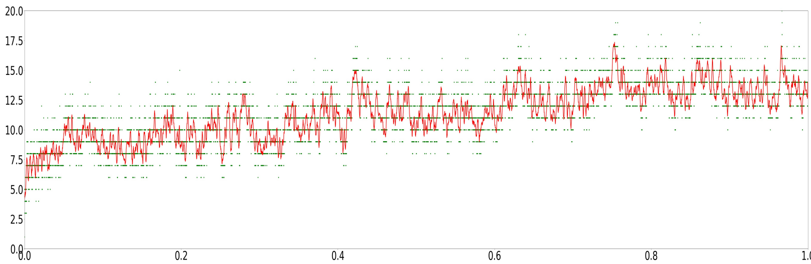
The most physical choice for ξ is that of a linear function $\xi(u) = \lambda u$ with some $\lambda \geq 0$. In this case the break-up mechanism for an empirical particle depends only on its size. As a result, λ controls the strength of the break-up mechanism.

Simulation Results

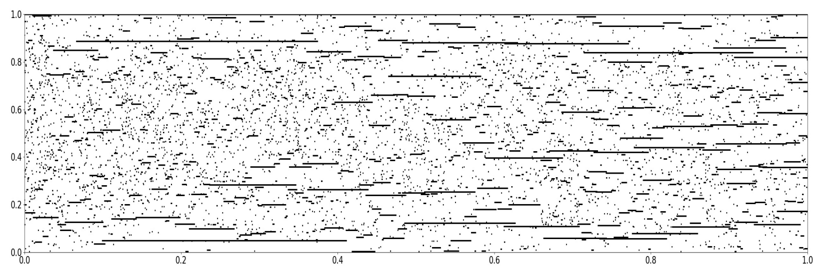
Below is a simulation of the empirical measure process μ_t, $t \geq 0$, for $\xi = \mathrm{id}$ starting from $\mu_0 = \delta_0$. Grayscale colour coding is for atom sizes. The red line is the center of mass of the system which is always a standard Brownian motion regardless the choice of ξ.



We also show the trajectory of the total number of atoms. The red curve shows a mollified (moving average) version of the same plot for better visibility.



Finally we plot the cooresponding history of induced partitions of the unit interval $[0, 1]$, where a dot represents the common boundary of two adjacent compartments belonging to two neighbouring atoms of μ.



1.3 Rigorous statement of main results

We will present now our main result in a rigorous fashion in terms of the measure valued process $\mu_.$ assuming values in the set $\mathcal{P}_2(\mathbb{R})$ of Borel probability measures on the real line with finite second moment, i.e.

$$\mathcal{P}_2(\mathbb{R}) := \{\rho \in \mathcal{P}(\mathbb{R}) : \; m_2(\rho) := \int_{\mathbb{R}} x^2 \rho(dx) < \infty\}.$$

We equip $\mathcal{P}_2(\mathbb{R})$ with the topology w_2 of weak convergence under uniform bounded second moment condition, i.e.

$$\rho_n \longrightarrow_{w_2} \rho :\Leftrightarrow \begin{cases} \rho_n \to \rho \;\; \text{weakly and} \\ m_2(\rho_n) \to m_2(\rho). \end{cases}$$

The free parameter of the model is given in terms of some $\eta \in \mathcal{P}_2(\mathbb{R})$, or equivalently by the choice of $\xi = g_\eta$, where for $\rho \in \mathcal{P}_2(\mathbb{R})$ we denote by g_ρ its right continuous quantile function, i.e

$$[0, 1] \ni u \mapsto g_\rho(u) := \inf\{x \in \mathbb{R} : \rho((-\infty, x]) > u\}.$$

Given $\eta \in \mathcal{P}_2(\mathbb{R})$ we introduce the set of all monotone transformations of η, i.e.

$$\mathcal{P}_2^\eta(\mathbb{R}) := \{\rho \in \mathcal{P}_2(\mathbb{R}) : \; \rho = h_\#(\eta) \text{ for some non decreasing } h : \mathbb{R} \mapsto \mathbb{R}\},$$

which is a w_2-closed subset of $\mathcal{P}_2(\mathbb{R})$. Finally, we write

$$\mathcal{P}_2^a(\mathbb{R}) = \left\{ \rho = \sum_{k=1}^n a_k \delta_{z_k} : \sum_{k=1}^n a_k = 1,\ a_k > 0,\ z_k \in \mathbb{R},\ k = 1, \ldots, n,\ n \in \mathbb{N} \right\}$$

for the subset of purely countably atomic probability measures on $\mathbb{R}$, and for $\rho \in \mathcal{P}_2^a(\mathbb{R})$ we set

$$|\rho| = \sum_{z \in \operatorname{supp} \rho} \delta_z \in \mathcal{P}_2(\mathbb{R}).$$

Below we will work with the algebra of ('smooth') functions $\mathcal{F}$ on $\mathcal{P}_2(\mathbb{R})$ which is generated by functions of the form

$$\begin{aligned} F(\rho) &= \phi(\langle h, g_\rho \rangle) \cdot \psi(\langle f, \rho \rangle) \\ &= \phi(\langle h, g_\rho \rangle) \cdot \psi(\langle f \circ g_\rho \rangle), \end{aligned}$$

where ϕ, ψ and h belong to $C_0^\infty(\mathbb{R})$, $f \in C^\infty([0,1])$ and $\langle \cdot, \cdot \rangle$ denotes the standard $L_2(dx)$ resp. duality product for measures vs. functions on $\mathbb{R}$ or $[0,1]$ and $\langle \cdot \rangle$ is integration against the uniform (Lebesgue) measure on $[0,1]$. Writing $F(\rho) = \Phi(g_\rho)$ for $F \in \mathcal{F}$ we define the *gradient* of $F \in \mathcal{F}$ by

$$\mathrm{D} F_{|\rho} := \operatorname{pr}_{g_\rho} \nabla^{L_2} \Phi_{|g_\rho},$$

where $\nabla^{L_2} \Phi$ denotes the standard $L_2(dx)$-gradient of Φ which is defined by

$$\langle \nabla^{L_2} \Phi_{|g}, h \rangle = \partial_{\varepsilon|\varepsilon=0} \Phi(g + \varepsilon h), \quad \forall h \in L_2[0,1],$$

and $\operatorname{pr}_{g_\rho}$ denotes the orthogonal projection in $L_2[0,1]$ onto the subspace of functions which are measurable with respect to the σ-field $\sigma(g_\rho)$ on $[0,1]$ generated by the function g_ρ. We will also use the projection $\operatorname{pr}_g^\perp$ to the complement, i.e. $\operatorname{pr}_g^\perp h = h - \operatorname{pr}_g h$.

With these preparations we can summarize the main result of this paper as follows.

Theorem 1 *For $\eta \in \mathcal{P}_2(\mathbb{R})$ there exists a measure Ξ^η on $\mathcal{P}^2(\mathbb{R})$ with* $\operatorname{supp} \Xi^\eta = \mathcal{P}_2^\eta(\mathbb{R})$ *such that the quadratic form*

$$\mathcal{E}(F,F) = \int_{\mathcal{P}_2^\eta(\mathbb{R})} \|\mathrm{D}_{|\rho} F(\cdot)\|^2_{L_2[0,1]} \, \Xi^\eta(d\rho), \quad F \in \mathcal{F},$$

is closable on $L_2(\mathcal{P}_2^\eta, \Xi^\eta)$, its closure being a local quasi-regular Dirichlet form on $L_2(\mathcal{P}_2^\eta, \Xi^\eta)$.
Let μ_t, $t \in [0, \zeta)$, the properly associated $\mathcal{P}_2^\eta(\mathbb{R})$-symmetric diffusion process with life time $\zeta > 0$. Then

i) for almost all $t \in [0, \zeta)$ it holds that $\mu_t \in \mathcal{P}_2^a$ almost surely;

ii) for all $f \in C_0^\infty(\mathbb{R})$ the process

$$M^f := \langle \mu_t, f\rangle - \frac{1}{2}\int_0^t \langle |\mu_s|, f''\rangle ds$$

is a local martingale with finite quadratic variation process

$$[M^f]_t = \int_0^t \langle \mu_s, (f')^2\rangle ds;$$

iii) for all $h \in C^\infty([0,1])$ the process

$$\tilde{M}^h := \langle g_{\mu_t}, h\rangle - \frac{1}{2}\int_0^t \langle \mathrm{pr}^\perp_{g_{\mu_s}} h, g_\eta\rangle ds$$

is a local martingale with finite quadratic variation process

$$[\tilde{M}^h]_t = \int_0^t \| \mathrm{pr}_{g_{\mu_s}} h\|^2_{L_2[0,1]} ds;$$

iv) for all measurable $A, B \subset \mathcal{P}_2^\eta$ with $0 < \Xi^\eta(A)\Xi^\eta(B) < \infty$ and A or B open it holds that

$$\lim_{t\to 0} t \cdot \ln \mathbb{P}(\mu_0 \in A, \mu_t \in B) = -\frac{d^2_{\mathcal{W}}(A,B)}{2},$$

where $d_{\mathcal{W}}(A,B) = \operatorname{ess\,inf}_{(\rho,\lambda)\in A\times B} d_{\mathcal{W}}(\rho,\lambda)$.

Remark 1 1) Property ii) in the theorem above is equivalent to saying that $\mu.$ is a martingale solution to the SPDE

$$d\mu_t = \Gamma_0(\mu_t)dt + \mathrm{div}\,(\sqrt{\mu_t} dW_t)$$

if one works with the canonical set of test functions of the type $\rho \mapsto \Phi(\rho) := \varphi(\langle f, \rho\rangle)$ with $\varphi, f \in C_0^\infty(\mathbb{R})$. This collection of test functions is commonly used in the theory of measure valued diffusion processes. Since ii) holds true regardless the choice of $\eta \in \mathcal{P}_2(\mathbb{R})$, it is clearly not sufficient to characterize the process $\mu.$. This shows in particular that the martingale problem encoded by ii) alone is not well posed. For instance, the solution given by the modified Arratia flow in [35] is obtained by choosing $\eta = \delta_z$ for some $z \in \mathbb{R}$, which, however, is not reversible.

2) In fact, property ii) will be a rather straightforward consequence of the stronger assertion iii), which is equivalent to the statement that process $X_t := g_{\mu_t}$, $t \in [0,\zeta)$, is a weak solution to the SDE in infinite dimensions

$$dX_t = \frac{1}{2}\,\mathrm{pr}^\perp_{X_t} \xi\, dt + \mathrm{pr}_{X_t} dW_t,$$

where $\xi = g_\eta$ and dW is $L_2[0,1]$-white noise. This representation is the justification for the heuristic description of the model in the previous section. As discussed in [35] the modified massive Arratia flow solves the same SDE with $\xi = \text{const.}$, i.e. $\eta = \delta_z$ for some $z \in \mathbb{R}$.

3) Property iii) together with the fact that $\operatorname{supp} \Xi^\eta = \mathcal{P}_2^\eta$ imply in particular that the process $\mu_\cdot$ explores the entire $\mathcal{P}_2^\eta$-space. Note that $\mathcal{P}_2^\eta = \mathcal{P}_2$ iff η has no atoms.

4) In Section 6 below we give a first condition assuring infinite lifetime $\zeta = \infty$. This will be the case if e.g. $\eta([a,b]) = 1$ for some $a \le b$ and $\eta(\{a\}) \cdot \eta(\{b\}) > 0$.

Remark 2 Our construction given in the subsequent sections is strongly related to diffusion processes on domains with so called sticky-reflecting boundary conditions. In fact, as in [46] we will cast the measure valued process $\mu_\cdot$ in terms of the associated process of quantile functions $X_\cdot = g_{\mu_\cdot}$, assuming values in the set $D^\uparrow$ of non decreasing functions on $[0,1]$. We view $D^\uparrow$ as a closed convex cone embedded in the topological space $L_2[0,1]$. As our main and critical step we construct the measure $\Xi = \Xi^\xi$ on $D^\uparrow$ which allows for an integration by parts formula to obtain a closable pre-Dirichlet form

$$\mathcal{E}(F,F) = \int_{D^\uparrow} \|\mathrm{D}F_{|g}\|_{L_2}^2 \Xi(dg).$$

As a subset of $L_2[0,1]$ the space $D^\uparrow$ has no interior since $\partial D^\uparrow$ is dense in $D^\uparrow$, hence we need a non-standard construction of a candidate measure Ξ. Our approach is to define Ξ on the subset $\mathcal{S}^\uparrow$ of piecewise constant non decreasing functions. The set $\mathcal{S}^\uparrow = \bigcup_{n=0}^\infty \mathcal{S}_n^\uparrow$ has a natural structure as a generalized non locally finite simplicial complex, where each $\mathcal{S}_n^\uparrow$ is the collection of all piecewise constant n-step functions. In this picture each connected component of the relative affine interior of $\mathcal{S}_n^\uparrow$ can be viewed as an n-dimensional face of $\mathcal{S}^\uparrow$ which is the common boundary of uncountably many $(n+1)$-dimensional faces that are parametrized by points in appropriate simplex. The measure Ξ^ξ is then obtained by putting an n-dimensonal measure Ξ_n^ξ on each $\mathcal{S}_n^\uparrow$ for all n in a way which is consistent with the hierarchical structure of $\mathcal{S}^\uparrow$. As a result we obtain a measure on a simplicial complex with positive mass on all faces of arbitrary dimension. In this picture the gradient operator appearing in the Dirichlet form above is obtained as projection of the full gradient to the effective tangent space on the respective faces and is therefore geometrically natural. The outcome is a Dirichlet form which generalizes the case considered e.g. in [26] to the (infinite dimensional) case of sticky-reflecting behaviour in piecewise smooth domains along embedded boundaries but now of arbitrary codimension.

The *structure of this work* is as follows. After some preliminaries we start off in Chapter 3 by introducing the model in a special case when the system consists of a fixed finite number of atoms with prescribed masses. The atoms can coalesce and fragmentate, but fragmentation is allowed only in accordance with the initially assigned mass portions. This chapter exhibits the basic mechanism of the system in a finite dimensional situation.

Section 4 contains the construction of the measure Ξ^{ξ} in the general case. We identify its support and show certain moment bounds which are critical for the quasi-regularity of the Dirichlet form which we introduce in Section 5. The core result of Section 5 is the integration by parts formula which is needed for closability. In Section 6 we establish quasi-regularity. We also show conservativeness in a special case. Section 7 is devoted to the identification of the intrinsic metric which leads to the desired Varadhan formula by applying a general theorem by Ariyoshi and Hino [5]. In Section 8 we wrap up the results in terms of the induced measure valued process and the related martingale problem.

2 Preliminaries

For $p \in [1, \infty]$ we denote the space of all p-integrable (essentially bounded if $p = \infty$) functions (more precisely equivalence classes) from $[0, 1]$ to $\mathbb{R}$ with respect to the Lebesgue measure Leb on $[0, 1]$ by L_p and $\|\cdot\|_p$ is the usual norm on L_p. The inner product in L_2 is denoted by $\langle\cdot,\cdot\rangle$. Let $D^{\uparrow}$ be the set of càdlàg non decreasing functions from $[0, 1]$ into $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. For convenience, we assume that all functions from $D^{\uparrow}$ are continuous at 1. Let $L_p^{\uparrow}$ be the subset of L_p that contains functions (their equivalence classes) from $D^{\uparrow}$.

Note that $L_2^{\uparrow}$ is a closed subset of L_2, by Corollary A.2. [31]. Consequently, $L_2^{\uparrow}$ is a Polish space with respect to the distance induced by $\|\cdot\|_2$.

If $f = g$ a.e., then we say that f is a *modification* or *version* of g or g is a *modification* or *version* of f.

Remark 3 Since each function f from $L_2^{\uparrow}$ has a unique modification from $D^{\uparrow}$ (see, e.g., Remark A.6. [31]), considering f as a map from $[0, 1]$ to $\overline{\mathbb{R}}$, we always take its modification from $D^{\uparrow}$.

We set for each $n \in \mathbb{N}$

$$E^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_i \leq x_{i+1}, \ i \in [n-1]\}$$

and

$$E_0^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_i < x_{i+1}, \ i \in [n-1]\},$$

where $[n] = \{1, \ldots, n\}$. Also let

$$Q^n = \{q = (q_1, \ldots, q_{n-1}) : \ 0 < q_1 < \ldots < q_{n-1} < 1\}$$

for all $n \geq 2$. For convenience, considering q from Q^n, we always set $q_0 = 0$ and $q_n = 1$.

Next, for $g \in L_2^{\uparrow}$ we denote the number of distinct values of the function g (that belongs to $D^{\uparrow}$ according to the previous remark) by $\sharp g$. If $\sharp g < \infty$, then g is called a *step function* (g takes a finite number of values). The set of all step functions we denote by $\mathcal{S}^{\uparrow}$.

Remark 4 If $\sharp g = n$, then there exist unique $q \in Q^n$ and $x \in E_0^n$ such that

$$g = \sum_{i=1}^{n} x_i \mathbb{I}_{[q_{i-1},q_i)} + x_n \mathbb{I}_{\{1\}},$$

where $\mathbb{I}_A$ is the indicator function of a set A.

If E is a topological space, then the Borel σ-algebra on E is denoted by $\mathcal{B}(E)$.

For any family of sets $\mathcal{H}$ we denote the smallest σ-algebra that contains $\mathcal{H}$ by $\sigma(\mathcal{H})$. Similarly, $\sigma(f) = \sigma(\{f^{-1}(A) : A \in \mathcal{B}(\mathbb{R})\}) = \{f^{-1}(A) : A \in \mathcal{B}(\mathbb{R})\}$ for a function f taking values in $\mathbb{R}$. For $g \in L_2^{\uparrow}$ let $\sigma^\star(g)$ denote the completion of the σ-algebra $\sigma(g)$ with respect to the Lebesgue measure on $[0,1]$ and pr_g be the orthogonal projection operator in L_2 on the closed linear subspace

$$L_2(g) := \{f \in L_2 : \ f \text{ is } \sigma^\star(g)\text{-measurable}\}.$$

By Lemma 1.25 [28], $\sigma^\star(g)$ and $L_2(g)$ are well-defined for each equivalence class g from $L_2^{\uparrow}$. Also we set $L_2^{\uparrow}(g) = L_2(g) \cap L_2^{\uparrow}$.

Remark 5 (i) For each $h \in L_2$ the function $\mathrm{pr}_g h$ coincides with the conditional expectation $\mathbb{E}(h|\sigma^\star(g))$ on the probability space $([0,1], \mathcal{L}([0,1]), \mathrm{Leb})$, where $\mathcal{L}([0,1])$ denotes the σ-algebra of Lebesgue measurable subsets of $[0,1]$.
(ii) For each $h \in L_2$, $\mathbb{E}(h|\sigma^\star(g)) = \mathbb{E}(h|\sigma(g))$ a.e.
(iii) The projection pr_g maps the space $L_2^{\uparrow}$ into $L_2^{\uparrow}$.

3 Finite system of sticky reflected diffusion particles

The aim of this section is to construct a finite system of diffusion particles on the real line with sticky-reflecting interaction. Also this section gives a motivation for the definition of the system in the general case. We will use a Dirichlet form approach for the construction of the system. In particular, we use ideas from paper [26] for the description of the sticky-reflecting mechanism. Here we fix $n \in \mathbb{N}$ and numbers $m_i \in (0,1]$, $i \in [n]$, with $m_1 + \ldots + m_n = 1$, which play a role of a number of particles and particle masses respectively.

3.1 Some notation

Let Θ^n denote the set of all ordered partitions of $[n]$. We take $\theta = (\theta_1, \ldots, \theta_p) \in \Theta^n$ and denote the number of sets in the partition θ by $|\theta|$, i.e. $|\theta| = p$. Let

$$E_\theta = \{x \in E^n : \ x_i = x_j \quad \Leftrightarrow \quad i,j \in \theta_k \ \text{ for some } \ k \in [|\theta|]\}.$$

Remark that the sets E_θ, $E_{\theta'}$ are disjoint for $\theta \neq \theta'$ and $E^n = \bigcup_{\theta \in \Theta^n} E_\theta$. Let R_θ be the bijection between E_θ and $E^{|\theta|}$ defined as follows

$$R_\theta\,(x_1, \ldots, x_n) = (y_1, \ldots, y_{|\theta|}),$$

where $y_k = x_i$ for some $i \in \theta_k$ (and, consequently, for all $i \in \theta_k$, since $x \in E_\theta$) and $k \in [|\theta|]$. The push forward of the Lebesgue measure $\lambda_{|\theta|}$ on $E^{|\theta|}$ under the map R_θ^{-1} is denoted by λ_θ. We note that λ_θ and $\lambda_{\theta'}$ are singular if $\theta \neq \theta'$. Let A_θ be the $n \times n$-matrix defined by

$$A_\theta = \operatorname{diag}\{A_{\theta_1}, \dots, A_{\theta_p}\},$$

where

$$A_{\theta_k} = \frac{1}{m_{\theta_k}} \begin{pmatrix} \sqrt{m_{i_k}} & \dots & \sqrt{m_{j_k}} \\ \dots & \dots & \dots \\ \sqrt{m_{i_k}} & \dots & \sqrt{m_{j_k}} \end{pmatrix}$$

for $\theta_k = \{i_k, \dots, j_k\}$, $i_k < \ldots < j_k$, and $m_{\theta_k} = \sum_{i \in \theta_k} m_i$, $k \in [|\theta|]$.

We say that $f : E^n \to \mathbb{R}$ belongs to $C_0^2(E^n)$ if it has a compact support and can be extended to a twice continuously differentiable function $\widetilde{f}$ on an open set that contains E^n. Set $\frac{\partial}{\partial x_i} f(x) := \frac{\partial}{\partial x_i} \widetilde{f}(x)$, $x \in E^n$, $i \in [n]$. Let

$$\nabla_\theta f(x) := \left(\frac{1}{\sqrt{m_{\theta_k}}} \frac{\partial}{\partial y_k} f(R_\theta^{-1}(y)) \big|_{y = R_\theta(x)} \right)_{k \in [|\theta|]}, \quad x \in E_\theta,$$

and

$$\triangle_\theta f(x) := \operatorname{Tr}\left(A_\theta A_\theta^T \nabla^2 f\right) = \sum_{k=1}^{|\theta|} \frac{1}{m_{\theta_k}} \frac{\partial^2}{\partial y_k^2} f(R_\theta^{-1}(y)) \big|_{y = R_\theta(x)}, \quad x \in E_\theta,$$

for $f \in C_0^2(E^n)$, where A^T denotes the transpose matrix.

3.2 Definition via Dirichlet forms

We define the measure Λ_n on E^n, that will play a role of an invariant measure for a system of particles, as follows

$$\Lambda_n = \sum_{\theta \in \Theta^n} c_\theta \lambda_\theta,$$

where c_θ, $\theta \in \Theta^n$, are positive constants that will be chosen later, and consider the following symmetric bilinear form on $L_2(E^n, \Lambda_n)$ defined on all functions f, g from $C_0^2(E^n)$

$$\begin{aligned} \mathcal{E}_n(f,g) &= \frac{1}{2} \sum_{\theta \in \Theta^n} \int_{E^n} \langle \nabla_\theta f(x), \nabla_\theta g(x) \rangle_{\mathbb{R}^{|\theta|}} \Lambda_n(dx) \\ &= \frac{1}{2} \sum_{\theta \in \Theta^n} c_\theta \int_{E^{|\theta|}} \left(\sum_{k=1}^{|\theta|} \frac{\partial}{\partial y_k} f(R_\theta^{-1}(y)) \frac{\partial}{\partial y_k} g(R_\theta^{-1}(y)) \frac{1}{m_{\theta_k}} \right) \lambda_{|\theta|}(dy), \end{aligned}$$

where $\langle x, y \rangle_{\mathbb{R}^p} = \sum_{k=1}^p x_k y_k$.

For each $\theta \in \Theta^n$ we denote

$$\partial\theta = \big\{\theta' \in \Theta^n : \ \theta' = (\theta_1, \ldots, \theta_{k-1}, \theta_k \cup \theta_{k+1}, \theta_{k+2}, \ldots, \theta_{|\theta|}) \\ \text{for some} \quad k \in [|\theta| - 1]\big\}$$

and define for $\theta' = (\theta'_j) \in \partial\theta$ the vector $b^{\theta,\theta'} \in \mathbb{R}^n$ as follows

$$b_i^{\theta,\theta'} = \begin{cases} -\frac{1}{m_{\theta_k}}, & i \in \theta_k, \\ \frac{1}{m_{\theta_{k+1}}}, & i \in \theta_{k+1}, \\ 0, & \text{otherwise}, \end{cases} \qquad i \in [n],$$

where k satisfies $\theta_k \cup \theta_{k+1} = \theta'_k$.

Using integration by parts formula, it is easily to prove the following statement.

Lemma 1 *For each $f, g \in C_0^2(E^n)$ the relation*

$$\mathcal{E}_n(f, g) = -\int_{E^n} L_n f(x) g(x) \Lambda_n(dx)$$

holds, where

$$L_n f(x) = \frac{1}{2} \sum_{\theta \in \Theta^n} \triangle_\theta f(x) \mathbb{I}_{E_\theta}(x) + \frac{1}{2} \sum_{\theta \in \Theta^n} \langle b^\theta, \nabla f(x) \rangle \mathbb{I}_{E_\theta}(x)$$

and

$$b^\theta = \frac{1}{c_\theta} \sum_{\tilde{\theta} : \theta \in \partial\tilde{\theta}} c_{\tilde{\theta}} b^{\tilde{\theta},\theta}.$$

It is obvious that $(L_n, C_0^2(E^n))$ is a non negative self-adjoint linear operator on $L_2(E^n, \Lambda_n)$. Consequently, the bilinear form $(\mathcal{E}_n, C_0^2(E^n))$ is closable, by Proposition I.3.3 [39]. We will denote its closure by $(\mathcal{E}_n, \mathbb{D}_n)$.

Theorem 2 *(i) The bilinear form $(\mathcal{E}_n, \mathbb{D}_n)$ is a densely defined, local, regular, conservative, symmetric Dirichlet form on $L_2(E^n, \Lambda_n)$.*

(ii) There exists a (Markov) diffusion[1] process

$$X^n = (\Omega^n, \mathcal{F}^n, \{\mathcal{F}_t^n\}_{t \geq 0}, \{X_t^n\}_{t \geq 0}, \{\mathbb{P}_x^n\}_{x \in E^n})$$

with state space E^n and infinite life time that is properly associated with $(\mathcal{E}_n, \mathbb{D}_n)$.

(iii) The process X^n is a weak solution in E^n of the stochastic differential equation

$$\begin{aligned} dX_t^n &= \sum_{\theta \in \Theta^n} A_\theta \mathbb{I}_{E_\theta}(X_t^n) dw(t) + \frac{1}{2} \sum_{\theta \in \Theta^n} b^\theta \mathbb{I}_{E_\theta}(X_t^n) dt, \\ X_0^n &= x \end{aligned} \tag{1}$$

[1] see Definition V.1.10 [39]

for $\mathcal{E}_n$-q.e. $x \in E^n$, where $w(t)$, $t \geq 0$, is an n-dimensional standard Wiener process.

Proof The proof of theorem follows from standard arguments (see e.g. Section 3 [26]). □

Choosing constants c_θ, $\theta \in \Theta^n$, by a special way, we can simplify equation (1). Let P_θ be the matrix defined similarly as A_θ with $\sqrt{m_i}$ replaced by m_i for all $i \in [n]$.

Remark 6 If the space $\mathbb{R}^n$ is furnished with the inner product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i m_i$, $x, y \in \mathbb{R}^n$, then the linear operator

$$x \to P_\theta x, \quad x \in \mathbb{R}^n,$$

is the orthogonal projection on $\mathbb{R}_\theta$, where $\mathbb{R}_\theta \subseteq \mathbb{R}^n$ is defined similarly as E_θ with E^n replaced by $\mathbb{R}^n$.

We also set $P_x := P_\theta$ for each $x \in E_\theta$.

Proposition 1 *Let $\varsigma \in E_0^n$. If*

$$c_\theta = \left(\prod_{k=1}^{|\theta|} m_{\theta_k}\right)\left(\prod_{k=1}^{|\theta|-1} (\varsigma_{i_k^\theta+1} - \varsigma_{i_k^\theta})\right), \quad \theta \in \Theta^n, \tag{2}$$

where $i_k^\theta = \max \theta_k$, then $b^\theta = \varsigma - P_\theta \varsigma$. Moreover, the process X is a weak solution in E^n of the stochastic differential equation

$$\begin{aligned} dX_t^n &= P_{X_t^n} dB(t) + \frac{1}{2}(\varsigma - P_{X_t^n}\varsigma)dt, \\ X_0^n &= x \end{aligned} \tag{3}$$

for $\mathcal{E}_n$-q.e. $x \in E^n$, where $B(t)$, $t \geq 0$, is an n-dimensional Wiener process with

$$\mathrm{Var}\,(B_i(t), B_j(t)) = \frac{t}{m_i}\mathbb{I}_{\{i=j\}}, \quad i, j \in [n].$$

Proof First we show that $b^\theta = \varsigma - P_\theta\varsigma$. So, let $\theta \in \Theta^n$ be fixed. We will suppose that $\theta \neq (\{i\})_{i\in[n]}$, since the case $\theta = (\{i\})_{i\in[n]}$ is trivial. We also fix $j \in [n]$ and take k such that $j \in \theta_k$.

Let

$$\underline{j} := \min \theta_k, \quad \overline{j} := \max \theta_k$$

and for each $l \in \{\underline{j}, \dots, \overline{j} - 1\}$ we denote the sets $\{\underline{j}, \dots, l\}$ and $\{l+1, \dots, \overline{j}\}$ by $\{\leq l\}$ and $\{> l\}$, respectively. Noting that $b_j^{\tilde\theta,\theta} = 0$ for all $\tilde\theta \in \Theta^n$ satisfying $\theta \in \partial\tilde\theta$ and $\tilde\theta_k \cup \tilde\theta_{k+1} \neq \theta_k$, it is easily seen that

$$b_j^\theta = \begin{cases} \frac{1}{c_\theta}\sum_{l=\underline{j}}^{\overline{j}-1} c_{\theta^l} b_j^{\theta^l,\theta}, & \underline{j} < \overline{j}, \\ 0, & \underline{j} = \overline{j}, \end{cases}$$

where $\theta \in \partial\theta^l$ with $\theta^l_k = \{\leq l\}$ and $\theta^l_{k+1} = \{> l\}$. We assume that $\underline{j} < \overline{j}$, otherwise $b^\theta_j = \varsigma_j - (P_\theta \varsigma)_j = 0$. The simple computation gives

$$\frac{c_{\theta^l}}{c_\theta} = \frac{m_{\{\leq l\}} m_{\{>l\}}}{m_{\theta_k}} (\varsigma_{l+1} - \varsigma_l)$$

and

$$b^{\theta^l,\theta}_j = \begin{cases} -\frac{1}{m_{\{\leq l\}}}, & l \geq j, \\ \frac{1}{m_{\{>l\}}}, & l < j, \end{cases}$$

for all $l \in \{\underline{j}, \ldots, \overline{j} - 1\}$. Hence,

$$\begin{aligned} b^\theta_j &= \frac{1}{m_{\theta_k}} \left[\sum_{l=\underline{j}}^{j-1} m_{\{\leq l\}} (\varsigma_{l+1} - \varsigma_l) - \sum_{l=j}^{\overline{j}-1} m_{\{>l\}} (\varsigma_{l+1} - \varsigma_l) \right] \\ &= \frac{1}{m_{\theta_k}} \left[m_{\{\leq j-1\}} \varsigma_j - \sum_{l=\underline{j}}^{j-1} m_l \varsigma_l + m_{\{>j-1\}} \varsigma_j - \sum_{l=j}^{\overline{j}} m_l \varsigma_l \right] \\ &= \varsigma_j - \frac{1}{m_{\theta_k}} \sum_{l=\underline{j}}^{\overline{j}} m_l \varsigma_l = \varsigma_j - (P_\theta \varsigma)_j. \end{aligned}$$

Thus, $b^\theta = \varsigma - P_\theta \varsigma$.

The equality of the diffusion parts of (1) and (3) is trivial for $B_i(t) = \frac{w_i(t)}{\sqrt{m_i}}$, $i \in [n]$. The proposition is proved. □

The following example shows that one cannot expect that equation (3) has a strong solution.

Example 1 Let $n = 2$, $m_1 = m_2 = \frac{1}{2}$ and $\varsigma = (0, 1)$. Then $X_t = (x_1(t), x_2(t))$, $t \geq 0$, solves the equation

$$\begin{aligned} dx_1(t) &= \sqrt{2} \mathbb{I}_{\{x_1(t) \neq x_2(t)\}} dw_1(t) \\ &\quad + \mathbb{I}_{\{x_1(t) = x_2(t)\}} \frac{dw_1(t) + dw_2(t)}{\sqrt{2}} - \frac{1}{4} \mathbb{I}_{\{x_1(t) = x_2(t)\}} dt, \\ dx_2(t) &= \sqrt{2} \mathbb{I}_{\{x_1(t) \neq x_2(t)\}} dw_2(t) \\ &\quad + \mathbb{I}_{\{x_1(t) = x_2(t)\}} \frac{dw_1(t) + dw_2(t)}{\sqrt{2}} + \frac{1}{4} \mathbb{I}_{\{x_1(t) = x_2(t)\}} dt, \end{aligned}$$

where (w_1, w_2) is a 2-dimensional standard Wiener process. Taking

$$y_1(t) = \frac{x_2(t) - x_1(t)}{2} \quad \text{and} \quad y_2(t) = \frac{x_2(t) + x_1(t)}{2}, \quad t \geq 0,$$

it is easily seen that y_1 and y_2 are weak solutions of the equations

$$\begin{aligned} dy_1(t) &= \mathbb{I}_{\{y_1(t) > 0\}} d\tilde{w}_1(t) + \frac{1}{4} \mathbb{I}_{\{y_1(t) = 0\}} dt, \\ dy_2(t) &= d\tilde{w}_2(t). \end{aligned}$$

But the equation for y_1 has no strong solution, according to [20].

4 σ-finite measure on $L_2^{\uparrow}$

Now we want to transfer the results obtained in the previous section on the space of all non increasing functions and construct a process on the space $L_2^{\uparrow}$ that is similar to the process defined in Proposition 1, since in this case we have a good description for the drift term. First of all, we need a measure on $L_2^{\uparrow}$ that will play a role of invariant measure for the system of sticky reflected particles for any non decreasing function ξ (instead of the vector ς). Moreover, this measure should coincide with the measure Λ_n (where c_θ, $\theta \in \Theta^n$, are defined by (2)) for a finite number of particles. The construction of such a measure is the aim of this section.

Hereinafter $\xi \in D^{\uparrow}$ is a fixed bounded function.

4.1 Motivation of the definition

Here we will make some manipulations with the measure Λ_n in order to guess the needed measure. Let $n \in \mathbb{N}$, $m_i = \frac{i}{n}$, $i \in [n]$, and the constants c_θ from the definition of Λ_n be defined by (2) for some ς that will be chosen later. We transfer the measure

$$\Lambda_n = \sum_{\theta\in\Theta^n} \left(\prod_{k=1}^{|\theta|} m_{\theta_k}\right) \left(\prod_{k=1}^{|\theta|-1} (\varsigma_{i_k^\theta+1} - \varsigma_{i_k^\theta})\right) \lambda_\theta$$

on $L_2^{\uparrow}$ by the map

$$x \mapsto G(x) = \sum_{i=1}^{n} x_i \mathbb{I}_{\left[\frac{i-1}{n}, \frac{i}{n}\right)}, \quad x \in E^n.$$

So, let $\widetilde{\Lambda}_n$ be the push forward of the measure Λ_n on E^n under the map G. Then $\widetilde{\Lambda}_n$ can be written as follows

$$\widetilde{\Lambda}_n = \sum_{\theta\in\Theta^n} \left(\prod_{k=1}^{|\theta|} m_{\theta_k}\right) \left(\prod_{k=1}^{|\theta|-1} (\varsigma_{i_k^\theta+1} - \varsigma_{i_k^\theta})\right) \widetilde{\lambda}(m_{\theta_1}, \dots, m_{\theta_{|\theta|}}),$$

where $\widetilde{\lambda}(m_{\theta_1}, \dots, m_{\theta_{|\theta|}})$ is the push forward of the Lebesgue measure $\lambda_{|\theta|}$ on $E^{|\theta|}$ under the map $x \mapsto \sum_{k=1}^{|\theta|} x_k \mathbb{I}_{[a_{k+1}, a_k)}$, with $a_0 = 0$, $a_k = m_{\theta_k} + a_{k-1}$, $k \in [|\theta|]$.

Setting $\Theta_p^n = \{\theta \in \Theta^n : \ |\theta| = p\}$ and $\varsigma_{i+1} - \varsigma_i \approx \frac{1}{n}\xi'\left(\frac{i}{n}\right)$ (if ξ is continuously differentiable), it is easy to see that

$$\begin{aligned}
\widetilde{\Lambda}_n &= \sum_{p=1}^{n} \sum_{\theta\in\Theta_p^n} \left[\prod_{k=1}^{p} \frac{|\theta_k|}{n}\right] \left[\prod_{k=1}^{p-1} \xi'\left(\frac{i_k^\theta}{n}\right) \frac{1}{n}\right] \widetilde{\lambda}\left(\frac{|\theta_1|}{n}, \dots, \frac{|\theta_p|}{n}\right) \\
&= \sum_{p=1}^{n} \sum_{\substack{l_1,\dots,l_p \geq 1 \\ l_1+\dots+l_p = n}} \left[\prod_{k=1}^{p} \frac{l_k}{n}\right] \frac{1}{n^{p-1}} \left[\prod_{k=1}^{p-1} \xi'\left(\frac{l_1+\dots+l_k}{n}\right)\right] \widetilde{\lambda}\left(\frac{l_1}{n}, \dots, \frac{l_p}{n}\right).
\end{aligned}$$

Thus, we see that the relation consist of Riemann sums. So, we replace the measure $\widetilde{\Lambda}_n$ by

$$\sum_{p=1}^{n} \int_{\substack{r_1,\dots,r_{p-1}>0 \\ r_1+\dots+r_{p-1}<1}} \left(\prod_{k=1}^{p-1} r_k\right) (1-r_1-\dots-r_{p-1}) \\ \cdot \left(\prod_{k=1}^{p-1} \xi'(r_1+\dots+r_k)\right) \widetilde{\lambda}(r_1,\dots,r_{p-1},1-r_1-\dots-r_{p-1})\, dr$$

$$= \sum_{p=1}^{n} \int_{0<q_1<\dots<q_{p-1}<1} \left(\prod_{k=1}^{p}(q_k-q_{k-1})\right) \left(\prod_{k=1}^{p-1} \xi'(q_k)\right) \widetilde{\lambda}(q_1,q_2-q_1\dots,1-q_{p-1})\, dq$$
$$= \sum_{p=1}^{n} \int_{0<q_1<\dots<q_{p-1}<1} \left(\prod_{k=1}^{p}(q_k-q_{k-1})\right) \widetilde{\lambda}(q_1,q_2-q_1\dots,1-q_{p-1})\, d\xi^{\otimes(p-1)}(q),$$

where $q_0 = 0$ and $q_p = 1$ in the product.

In the next section we will use the obtained expression for the definition of the needed measure.

4.2 Definition of an invariant measure on $L_2^{\uparrow}$

First we define a measure Ξ_n on L_2 for each $n \in \mathbb{N}$, supported on step functions with at most $n-1$ jumps. Let $\chi_n : Q^n \times E^n \to L_2^{\uparrow}$ with

$$\chi_n(q,x) = \sum_{i=1}^{n} x_i \mathbb{I}_{[q_{i-1},q_i)} + x_n \mathbb{I}_{\{1\}}, \quad x \in E^n, \quad q \in Q^n, \tag{4}$$

and

$$\chi_1(x) = x\mathbb{I}_{[0,1]}, \quad x \in \mathbb{R}.$$

Denote for all $q \in Q^n$, $n \geq 2$, the push forward of the Lebesgue measure λ_n on E^n under the map $\chi_n(q,\cdot)$ by $\nu_n(q,\cdot)$, i.e.

$$\nu_n(q,A) = \lambda_n\{x : \ \chi_n(q,x) \in A\}, \quad A \in \mathcal{B}(L_2^{\uparrow}),$$

and set

$$\Xi_n(A) = \int_{Q^n} \left(\prod_{i=1}^{n}(q_i-q_{i-1})\right) \nu_n(q,A) d\xi^{\otimes(n-1)}(q), \quad A \in \mathcal{B}(L_2^{\uparrow}),$$

where $\int_{Q^n} \dots d\xi^{\otimes(n-1)}(q)$ is the $(n-1)$-dim Lebesgue-Stieltjes integral with respect to $\xi^{\otimes(n-1)}(q) = \xi(q_1) \cdot \ldots \cdot \xi(q_{n-1})$. We also set

$$\Xi_1(A) = \lambda_1 \{x \in \mathbb{R} : \ \chi_1(x) \in A\}, \quad A \in \mathcal{B}(L_2^{\uparrow}). \tag{5}$$

Now we define the measure on $L_2^{\uparrow}$, that will be used for the definition of the Dirichlet form in the general case, as a sum of $\varXi_n$, that is,

$$\varXi := \sum_{n=1}^{\infty} \varXi_n = \varXi_1 + \sum_{n=2}^{\infty} \int_{Q^n} \left(\prod_{i=1}^{n} (q_i - q_{i-1}) \right) \nu_n(q, \cdot) d\xi^{\otimes(n-1)}(q).$$

Remark 7 If $\xi = \chi_n(q, \varsigma)$ for some $q \in Q^n$ and $\varsigma \in E_0^n$, then a simple calculation shows that $\varXi$ coincides with the push forward of the measure $\varLambda_n$ on E^n, defined in Section 3.2, under the map $x \mapsto \chi_n(q, x)$, $x \in E^n$, for $m_i = q_i - q_{i-1}$, $i \in [n]$, and c_θ, $\theta \in \varTheta^n$, given by (2).

4.3 Some properties of the measure $\varXi$

In this section, we prove some properties of the measures $\varXi$ and $\varXi_n$, $n \geq 1$. Define on Q^n the measure μ_ξ^n as follows

$$\mu_\xi^n(A) = \int_A \prod_{i=1}^{n} (q_i - q_{i-1}) d\xi^{\otimes(n-1)}(q), \quad A \in \mathcal{B}(Q^n), \quad n \geq 2.$$

Lemma 2 *For each $n \in \mathbb{N}$,*

(i) $\varXi_n$ is the push forward of the measure $\mu_\xi^n \otimes \lambda_n$ under the map χ_n, if $n \geq 2$;

(ii) $\varXi_n$ is σ-finite on $L_2^{\uparrow}$ and

$$\varXi_n(B_r) \leq \frac{2\pi^{\frac{n}{2}} r^n}{n! \varGamma\left(\frac{n}{2}\right)} (\xi(1) - \xi(0))^{n-1},$$

where $B_r = \{g \in L_2^{\uparrow} : \|g\|_2 \leq r\}$;

(iii) $\varXi_n(\{g \in L_2^{\uparrow} : \sharp g \neq n\}) = 0$, where $\sharp g$ denotes the number of distinct values of g (see Section 2).

Remark 8 We note that $\{g \in L_2^{\uparrow} : \sharp g \neq n\} \in \mathcal{B}(L_2^{\uparrow})$, since $\{g \in L_2^{\uparrow} : \sharp g \leq n\}$ is closed in $L_2^{\uparrow}$.

Remark 9 Property (ii) of Lemma 2 immediately implies that $\varXi$ is a σ-finite measure on $L_2^{\uparrow}$ with $\varXi(B_r) < \infty$.

Proof (Proof of Lemma 2) (i) follows from the definition of the measure $\varXi_n$ and Fubini's theorem.

The equality $\nu_n(q, \{g \in L_2^{\uparrow} : \sharp g \neq n\}) = 0$, for all $q \in Q^n$, implies (iii).

Let us note that (ii) is obvious for $n = 1$. We prove (ii) for $n \geq 2$. Let $q \in Q^n$ be fixed. We first estimate

$$\begin{aligned} \nu_n(q, B_r) &= \lambda_n \left\{ x \in E^n : \|\chi_n(q, x)\|_2^2 \leq r^2 \right\} \\ &= \lambda_n \left\{ x \in E^n : \sum_{i=1}^{n} x_i^2 (q_i - q_{i-1}) \leq r^2 \right\} \\ &\leq \frac{2\pi^{\frac{n}{2}} r^n}{n \varGamma\left(\frac{n}{2}\right)} \frac{1}{\sqrt{\prod_{i=1}^{n} (q_i - q_{i-1})}}. \end{aligned} \tag{6}$$

Here $\lambda_n\left\{x\in E^n:\ \sum_{i=1}^n x_i^2(q_i-q_{i-1})\le r^2\right\}$ is estimated by the n-dimensional volume of the ellipsoid $\sum_{i=1}^n x_i^2(q_i-q_{i-1})\le r^2$. Thus,

$$\begin{aligned}\Xi_n(B_r)&\le\frac{2\pi^{\frac{n}{2}}r^n}{n\Gamma\left(\frac{n}{2}\right)}\int_{Q^n}\sqrt{\prod_{i=1}^n(q_i-q_{i-1})}d\xi^{\otimes(n-1)}(q)\\&\le\frac{2\pi^{\frac{n}{2}}r^n}{n\Gamma\left(\frac{n}{2}\right)}\int_{Q^n}1d\xi^{\otimes(n-1)}(q)=\frac{2\pi^{\frac{n}{2}}r^n}{n!\Gamma\left(\frac{n}{2}\right)}(\xi(1)-\xi(0))^{n-1},\end{aligned}$$

where $n!$ is obtaining by symmetry. The lemma is proved. □

The following lemma is important for the proof of the quasi-regularity of the Dirichlet form in Section 6.1.

Lemma 3 *Let $C>0$, $q\in[1,2]$, $p,r\in[2,\infty)$ and $l\in[1,\infty)$ such that $\frac{l}{r}+\frac{2}{q}-\frac{l}{p}\le\frac{3}{2}$ and $r\le p$. Then there exists a constant $\tilde{C}$ which only depends on C and l such that*

$$\sup_{h\in H}\int_{L_2^\uparrow}\|g\|_p^l\|\operatorname{pr}_g h\|_2^2\,\mathbb{I}_{\{\|g\|_r\le C\}}\,\Xi(dg)\le\tilde{C},$$

where $H=\{h\in L_2:\ \|h\|_q\le 1\}$.

Proof First we estimate $\int_{B_C}\|g\|_p^l\|\operatorname{pr}_g h\|_2^2\,\Xi_n(dg)$ for each $n\ge 2$ and $\|h\|_q\le 1$, where $B_C=\{g\in L_2^\uparrow:\ \|g\|_r\le C\}$.

So, by the definition of Ξ_n, we have

$$\begin{aligned}&\int_{B_C}\|g\|_p^l\|\operatorname{pr}_g h\|_2^2\,\Xi_n(dg)=\int_{Q^n}\prod_{i=1}^n(q_i-q_{i-1})\\&\cdot\left[\int_{E^n}\left(\sum_{i=1}^n|x_i|^p(q_i-q_{i-1})\right)^{\frac{l}{p}}\left\|\operatorname{pr}_{\chi_n(q,x)}h\right\|_2^2\mathbb{I}_{B_C}(\chi_n(q,x))\lambda_n(dx)\right]d\xi^{\otimes(n-1)}(q).\end{aligned}$$

Next, let $(q,x)\in Q^n\times E^n$ and $\chi_n(q,x)\in B_C$. Then

$$\|\chi_n(q,x)\|_r^r=\sum_{i=1}^n|x_i|^r(q_i-q_{i-1})\le C^r.$$

Thus, $|x_i|\le\frac{C}{(q_i-q_{i-1})^{\frac{1}{r}}}$, $i\in[n]$, and, consequently,

$$\|\chi_n(q,x)\|_p^p=\sum_{i=1}^n|x_i|^p(q_i-q_{i-1})\le C^p\sum_{i=1}^n(q_i-q_{i-1})^{1-\frac{p}{r}}. \tag{7}$$

Similarly, if $\|\operatorname{pr}_{\chi_n(q,x)}h\|_q\le 1$, then

$$\|\operatorname{pr}_{\chi_n(q,x)}h\|_2^2\le\sum_{i=1}^n(q_i-q_{i-1})^{1-\frac{2}{q}}. \tag{8}$$

We note that by Remark 5 (i) and Jensen's inequality, we have that $\|h\|_q \leq 1$ implies $\| \operatorname{pr}_g h\|_q \leq 1$. Indeed,

$$\| \operatorname{pr}_g h\|_q^q = \mathbb{E}\,|\mathbb{E}(h|\sigma^\star(g))|^q \leq \mathbb{E}\mathbb{E}(|h|^q|\sigma^\star(g)) = \mathbb{E}|h|^q = \|h\|_q^q \leq 1.$$

So, (8) holds for any $h \in H$. Hence, using $q_i - q_{i-1} \leq 1$, $i \in [n]$, (8) and (7), we can estimate for each $h \in H$

$$\begin{aligned}
&\prod_{i=1}^n (q_i - q_{i-1}) \left(\sum_{i=1}^n |x_i|^p (q_i - q_{i-1}) \right)^{\frac{l}{p}} \left\| \operatorname{pr}_{\chi_n(q,x)} h \right\|_2^2 \mathbb{I}_{B_C}(\chi_n(q,x)) \\
&\quad \leq C^l \prod_{i=1}^n (q_i - q_{i-1}) \left(\sum_{i=1}^n (q_i - q_{i-1})^{1-\frac{p}{r}} \right)^{\frac{l}{p}} \left(\sum_{i=1}^n (q_i - q_{i-1})^{1-\frac{2}{q}} \right) \\
&\quad \leq C^l n^{\frac{l}{p}} \prod_{i=1}^n (q_i - q_{i-1})^{\frac{1}{2}} \left(\sum_{i=1}^n (q_i - q_{i-1})^{\frac{3}{2} - \frac{l}{r} - \frac{2}{q} + \frac{l}{p}} \right) \\
&\quad \leq C^l n^{\frac{l}{p}+1} \prod_{i=1}^n (q_i - q_{i-1})^{\frac{1}{2}} \mathbb{I}_{B_C}(\chi_n(q,x)),
\end{aligned}$$

if $\frac{l}{r} + \frac{2}{q} - \frac{l}{p} \leq \frac{3}{2}$ and $r \leq p$. Hence, by (6) and the inclusion $B_C \subseteq \{g \in L_2^\uparrow : \|g\|_2 \leq C\}$, $r \geq 2$, we have

$$\begin{aligned}
&\int_{B_C} \|g\|_p^l \| \operatorname{pr}_g h\|_2^2 \, \varXi_n(dg) \\
&\quad \leq C^l n^{\frac{l}{p}+1} \int_{Q^n} \prod_{i=1}^n (q_i - q_{i-1})^{\frac{1}{2}} \left[\int_{E^n} \mathbb{I}_{B_C}(\chi_n(q,x)) \lambda_n(dx) \right] d\xi^{\otimes(n-1)}(q) \\
&\quad = C^l n^{\frac{l}{p}+1} \int_{Q^n} \prod_{i=1}^n (q_i - q_{i-1})^{\frac{1}{2}} \nu_n(q, B_C) d\xi^{\otimes(n-1)}(q) \\
&\quad \leq \frac{2\pi^{\frac{n}{2}} C^{(n+l)} n^{\frac{l}{p}+1}}{n! \varGamma\left(\frac{n}{2}\right)} (\xi(1) - \xi(0))^{n-1}.
\end{aligned}$$

We note that $\sum_{n=2}^\infty \frac{2\pi^{\frac{n}{2}} C^{(n+l)} n^{\frac{l}{p}+1}}{n! \varGamma\left(\frac{n}{2}\right)} (\xi(1) - \xi(0))^{n-1} < \infty$ and

$$\sup_{h \in H} \int_{B_C} \|g\|_p^l \| \operatorname{pr}_g h\|_2^2 \, \varXi_1(dg) \leq \int_{-C}^C |x|^l dx,$$

since $\|g\|_p = \|g\|_2$ and $\| \operatorname{pr}_g h\|_2 = \| \operatorname{pr}_g h\|_q \leq \|h\|_q \leq 1$ $\varXi_1$-a.e. Hence, the integral $\int_{B_C} \|g\|_p^l \| \operatorname{pr}_g h\|_2^2 \, \varXi(dg)$ is uniformly bounded on H by a constant that only depends on l and C. The lemma is proved. □

Lemma 4 $\varXi \left\{ g \in L_2^\uparrow : \ \|g\|_p^p \not\to \|g\|_2^2 \ \ \textit{as} \ \ p \downarrow 2 \right\} = 0.$

Proof The proof immediately follows from the definition of the measure Ξ and the fact that for all $n \geq 2$ and $q \in Q^n$,

$$\nu_n\left(q, \left\{\chi_n(q,x) :\ x \in E^n \ \text{ and } \ \|\chi_n(q,x)\|_p^p \not\to \|\chi_n(q,x)\|_2^2,\ p \downarrow 2\right\}\right)$$
$$= \lambda_n\left\{x \in E^n :\ \sum_{i=1}^n x_i^p(q_i - q_{i-1}) \not\to \sum_{i=1}^n x_i^2(q_i - q_{i-1}),\ p \downarrow 2\right\} = 0.$$

□

4.4 Support of the measure Ξ

Recall that $L_2^{\uparrow}(\xi)$ denotes the subset of all $\sigma^\star(\xi)$-measurable functions from $L_2^{\uparrow}$. Let μ_ξ denote the Lebesgue-Stieltjes measure on $[0,1]$ generated by the function ξ, that is, $\mu_\xi((a,b]) = \xi(b) - \xi(a)$ for all $a < b$ from $[0,1]$.

Proposition 2 *The support of Ξ coincides with $L_2^{\uparrow}(\xi)$.*

Remark 10 If ξ is a strictly increasing function, then $L_2^{\uparrow}(\xi) = L_2^{\uparrow}$ and, consequently, $\operatorname{supp} \Xi = L_2^{\uparrow}$.

To prove Proposition 2, we need several auxiliary lemmas.

Lemma 5 *If $h \in \mathcal{S}^{\uparrow} \cap L_2^{\uparrow}(\xi)$ and s is a jump point of h, then $s \in \operatorname{supp} \mu_\xi$.*

Proof Suppose that $s \notin \operatorname{supp} \mu_\xi$. Then there exists $\varepsilon > 0$ such that $\mu_\xi((s - \varepsilon, s + \varepsilon]) = 0$. So, $\xi(s - \varepsilon) = \xi(s + \varepsilon)$. By Proposition 14, we have that $h(s - \varepsilon) = h((s + \varepsilon)-)$. But this contradicts the assumption that s is a jump point of the non decreasing function h. □

Lemma 6 *Let $g, h \in L_2^{\uparrow}$ and h is a step function. Then $\operatorname{pr}_g h$ is also a step function.*

Proof The proof if given in the appendix. □

Proof (Proof of Proposition 2) Step I. First we show that $L_2^{\uparrow}(\xi) \subseteq \operatorname{supp} \Xi$.

Let $g \in L_2^{\uparrow}(\xi)$ and $\varepsilon > 0$. We need to show that $\Xi(B_\varepsilon(g)) > 0$, where $B_\varepsilon(g) = \{h \in L_2^{\uparrow} :\ \|g - h\|_2 < \varepsilon\}$. Since the set of all step functions $\mathcal{S}^{\uparrow}$ is dense in $L_2^{\uparrow}$, there exists $\widetilde{h} \in \mathcal{S}^{\uparrow}$ such that $\|g - \widetilde{h}\|_2 < \varepsilon$. Hence,

$$\|g - \operatorname{pr}_\xi \widetilde{h}\|_2 = \|\operatorname{pr}_\xi(g - \widetilde{h})\|_2 \leq \|g - \widetilde{h}\|_2 < \varepsilon. \tag{9}$$

Setting $\overline{h} = \operatorname{pr}_\xi \widetilde{h}$ and using Lemma 6, we have that $\overline{h}$ is a step function that belongs to $B_\varepsilon(g) \cap L_2^{\uparrow}(\xi)$. By Remark 4, there exist $n \in \mathbb{N}$, $r \in Q^n$ (if $n \geq 2$) and $y \in E_0^n$ such that

$$\overline{h} = \sum_{i=1}^n y_i \mathbb{I}_{[r_{i-1}, r_i)} + y_n \mathbb{I}_{\{1\}}.$$

If $n=1$, then it is easy to see that $\Xi_1(B_\varepsilon(g))>0$. This implies $\Xi(B_\varepsilon(g))>0$. So, we give the proof for $n\geq 2$.

Using the continuity of the map $F: Q^n\times E_0^n\to\mathbb{R}$ given by

$$F(q,x)=\|g-\chi_n(q,x)\|_2^2=\sum_{i=1}^n\int_{q_{i-1}}^{q_i}(g(s)-x_i)^2ds,\quad (q,x)\in Q^n\times E_0^n,$$

where χ_n is defined by (4), and the inequality $F(r,y)<\varepsilon^2$ following from (9), we can conclude that there exist neighbourhoods of r and y given by

$$R=\{q\in Q^n:\ |q_i-r_i|<\delta,\ i\in[n-1]\},\quad Y=\{x\in\mathbb{R}^n:\ |x_i-y_i|<\delta,\ i\in[n]\}$$

such that $Y\subset E^n$, $\prod_{i=1}^n(q_i-q_{i-1})\geq\delta$ and $F(q,x)<\varepsilon^2$ for all $(q,x)\in R\times Y$. Thus, trivially, $\chi_n(q,x)\in B_\varepsilon(g)$ for all $(q,x)\in R\times Y$. So, we can estimate $\Xi_n(B_\varepsilon(g))$ from below as follows

$$\begin{aligned}\Xi_n(B_\varepsilon(g))&=\int_{Q^n}\prod_{i=1}^n(q_i-q_{i-1})\left(\int_{E_n}\mathbb{I}_{\{x:\ \chi_n(q,x)\in B_\varepsilon(g)\}}\lambda_n(dx)\right)d\xi^{\otimes(n-1)}(q)\\&\geq\delta\int_R\left(\int_Y 1\lambda_n(dx)\right)d\xi^{\otimes(n-1)}(q)=\delta^{n+1}\prod_{i=1}^{n-1}\mu_\xi((r_i-\delta,r_i+\delta)).\end{aligned}$$

Since $\overline{h}$ belongs to $\mathcal{S}^\uparrow\cap L_2^\uparrow(\xi)$ and r_i, $i\in[n-1]$, are its jump points,

$$\prod_{i=1}^{n-1}\mu_\xi((r_i-\delta,r_i+\delta))>0,$$

by Lemma 5. Hence $\Xi(B_\varepsilon(g))>0$ and consequently, $L_2^\uparrow(\xi)\subseteq\operatorname{supp}\Xi$.

Step II. Here we establish that for all $g\in L_2^\uparrow\setminus L_2^\uparrow(\xi)$ there exists $\varepsilon>0$ such that $\Xi(B_\varepsilon(g))=0$. Let $g\in L_2^\uparrow\setminus L_2^\uparrow(\xi)$ be fixed. Using Proposition 14, we can find $a,b\in[0,1]$ such that $\xi(a)=\xi(b)$ and $g(a)<g(b-)$. Thus, for some $\delta\in(0,b-a)$

$$g(a)<g(b-\delta)\leq g(b).$$

This inequality and the right continuity of g imply that g is not a constant a.e. on $[a,b]$.

Next we claim that there exists $\varepsilon>0$ such that

$$B_\varepsilon(g)\cap L_2^\uparrow\subseteq\{h\in L_2^\uparrow:\ h(a)<h(b)\}.\tag{10}$$

Indeed, let for a fixed $\varepsilon>0$ (which we will choose later) we can find h from $B_\varepsilon(g)\cap L_2^\uparrow$ that is a constant on $[a,b]$. Then

$$\begin{aligned}\|g-h\|_2^2&=\int_0^1(g(s)-h(s))^2ds\geq\int_a^b(g(s)-h(a))^2ds\\&\geq\int_a^b\left(g(s)-\frac{1}{b-a}\int_a^b g(r)dr\right)^2ds=\varepsilon_0>0,\end{aligned}$$

because g is not a constant a.e. on $[a,b]$. Hence, for $\varepsilon < \varepsilon_0$ the inclusion (10) holds.

Now we are ready to estimate $\Xi(B_\varepsilon(g))$ for any fixed $\varepsilon < \varepsilon_0$. So,

$$\Xi(B_\varepsilon(g)) = \Xi\left(\{h \in L_2^\uparrow : \ h(a) < h(b)\} \cap B_\varepsilon(g)\right)$$
$$= \sum_{n=2}^{\infty} \int_{Q^n} \prod_{i=1}^{n} (q_i - q_{i-1}) \nu_n \left(q, \{h \in L_2^\uparrow : \ h(a) < h(b)\} \cap B_\varepsilon(g)\right) d\xi^{\otimes(n-1)}(q).$$

Let $n \geq 2$ and

$$Q_{a,b}^n := \{q \in Q^n : \ q_i \notin (a,b] \ \text{ for all } \ i \in [n-1]\}.$$

Then for all $q \in Q_{a,b}^n$

$$\nu_n \left(q, \{h \in L_2^\uparrow : \ h(a) < h(b)\} \cap B_\varepsilon(g)\right) = 0,$$

since $\nu_n(q, \cdot)$ is supported on the set of step functions that have no jumps on $(a,b]$. Moreover, due to the inclusion $Q^n \setminus Q_{a,b}^n \subseteq \bigcup_{i=1}^{n-1} Q_{a,b,i}^n$, where $Q_{a,b,i}^n := \{q \in [0,1]^{n-1} : \ q_i \in (a,b]\}$, and the equality $\xi(a) = \xi(b)$, we have

$$\mu_\xi^n(Q^n \setminus Q_{a,b}^n) \leq \sum_{i=1}^{n-1} \mu_\xi^n(Q_{a,b,i}^n) = \int_{Q_{a,b,i}^n} \prod_{i=1}^{n} (q_i - q_{i-1}) d\xi^{\otimes(n-1)}(q)$$
$$\leq \sum_{i=1}^{n-1} (\xi(1) - \xi(0))^{n-2} (\xi(b) - \xi(a)) = 0.$$

Thus, $\Xi(B_\varepsilon(g)) = 0$. This finishes the proof of the proposition. □

Corollary 1 *If $\sharp\xi \geq n$, then* $\operatorname{supp} \Xi_n = L_2^\uparrow(\xi) \cap \{g \in L_2^\uparrow : \ \sharp g \leq n\}$. *Otherwise,* $\Xi_n = 0$.

Proof The inclusion $\operatorname{supp} \Xi_n \subseteq L_2^\uparrow(\xi) \cap \{g \in L_2^\uparrow : \ \sharp g \leq n\}$ immediately follows from Proposition 2 and Lemma 2 (*iii*).

Next assuming $\sharp\xi \geq n$, we prove that

$$\Xi_n(B_\varepsilon(g)) > 0 \tag{11}$$

for all $g \in L_2^\uparrow(\xi) \cap \{g \in L_2^\uparrow : \ \sharp g \leq n\}$ and $\varepsilon > 0$. Since the close of $\{g \in L_2^\uparrow : \sharp g = n\} \cap L_2^\uparrow(\xi)$ in $L_2^\uparrow$ coincides with $\{g \in L_2^\uparrow : \ \sharp g \leq n\} \cap L_2^\uparrow(\xi)$, it is enough to prove inequality (11) for functions of the form

$$g = \chi(q,x), \quad (q,x) \in Q^n \times E_0^n.$$

So, fixing $g = \chi(q,x)$ for some $(q,x) \in Q^n \times E_0^n$, similarly as in Step I of the proof of Proposition 2 we can show that $\Xi_n(B_\varepsilon(g)) > 0$. Hence, $\operatorname{supp} \Xi_n = L_2^\uparrow(\xi) \cap \{g \in L_2^\uparrow : \ \sharp g \leq n\}$.

If $\sharp\xi < n$, then $L_2^\uparrow(\xi) \cap \{g \in L_2^\uparrow : \ \sharp g = n\} = \emptyset$, by Proposition 14. Consequently, Proposition 2 together with Lemma 2 (*iii*) yield $\Xi_n = 0$. □

Corollary 2 *The set $\mathcal{S}^\uparrow \cap L_2^\uparrow(\xi)$ has full measure Ξ, that is, $\Xi(L_2^\uparrow(\xi) \setminus \mathcal{S}^\uparrow) = 0$.*

Proof The corollary follows from the definition of the measure Ξ and Corollary 1. □

5 Definition of the Dirichlet form in the general case

In this section we define the Dirichlet form in general case. As before, we will assume that $\xi \in D^{\uparrow}$ is a bounded function that generates the measure $\varXi$ on $L_2^{\uparrow}$. Since this measure is supported on the space $L_2^{\uparrow}(\xi)$, hereinafter we will work with spaces $L_2^{\uparrow}(\xi)$ and $L_2(\xi)$ instead of $L_2^{\uparrow}$ and L_2. Let $L_2(L_2^{\uparrow}(\xi), \varXi)$ or simpler $L_2(\varXi)$ denote the space of $\varXi$-integrable functions on $L_2^{\uparrow}$ with the usual norm $\|\cdot\|_{L_2(\varXi)}$ and the inner product $\langle \cdot, \cdot \rangle_{L_2(\varXi)}$.

5.1 A set of admissible functions on $L_2^{\uparrow}(\xi)$

Let $C_b^{\infty}(\mathbb{R}^m)$ denote the set of all infinitely differentiable (real-valued) functions on $\mathbb{R}^m$ with all partial derivatives bounded and $C_0^{\infty}(\mathbb{R}^m)$ be the set of functions from $C_b^{\infty}(\mathbb{R}^m)$ with compact support. In this section we want to define the class of "smooth" integrable functions on $L_2^{\uparrow}(\xi)$. Since $L_2^{\uparrow}(\xi) \subseteq L_2(\xi)$, it is reasonable to consider functions of the form $u(\langle \cdot, h_1 \rangle, \ldots, \langle \cdot, h_m \rangle)$, where $u \in C_b^{\infty}(\mathbb{R}^m)$ and $h_j \in L_2(\xi)$, $j \in [m]$. But in general, these functions are not integrable with respect to the measure $\varXi$. So, we will cut off they by functions with bounded support in $L_2^{\uparrow}(\xi)$. Let $\mathcal{FC}$ denote the linear space generated by functions on $L_2^{\uparrow}(\xi)$ of the form

$$U = u(\langle \cdot, h_1 \rangle, \ldots, \langle \cdot, h_m \rangle)\varphi(\|\cdot\|_2^2) = u(\langle \cdot, \mathbf{h} \rangle)\varphi(\|\cdot\|_2^2), \tag{12}$$

where $u \in C_b^{\infty}(\mathbb{R}^m)$, $\varphi \in C_0^{\infty}(\mathbb{R})$ and $h_j \in L_2(\xi)$, $j \in [m]$.

Remark 11 (i) The set $\mathcal{FC}$ is an associative algebra, in particular, $U, V \in \mathcal{FC}$ implies $UV \in \mathcal{FC}$.

(ii) Since each $U \in \mathcal{FC}$ has a bounded support, $\mathcal{FC} \subseteq L_2(L_2^{\uparrow}(\xi), \varXi)$, by Remark 9.

(iii) For each $n \geq 2$ and $q \in Q^n$ the function $x \mapsto U(\chi_n(q, x))$ belongs to $C_0^{\infty}(E^n)$ and, similarly, $x \mapsto U(\chi_1(x))$ belongs to $C_0^{\infty}(\mathbb{R})$.

Proposition 3 *The set $\mathcal{FC}$ is dense in $L_2(L_2^{\uparrow}(\xi), \varXi)$.*

Proof Let $\varphi_n \in C_0^{\infty}(\mathbb{R})$ take values from $[0, 1]$ and satisfy

$$\varphi_n(x) = \begin{cases} 1, & |x| \leq n^2 - 1, \\ 0, & |x| \geq n^2, \end{cases}$$

and let $U \in L_2(\varXi)$. Since

$$\begin{aligned} \|U - U\varphi_n(\|\cdot\|_2^2)\|_{L_2(\varXi)}^2 &= \int_{L_2^{\uparrow}(\xi)} U^2(g) \left(1 - \varphi_n\left(\|g\|_2^2\right)\right)^2 \varXi(dg) \\ &= \int_{B_n^c} U^2(g) \left(1 - \varphi_n\left(\|g\|_2^2\right)\right)^2 \varXi(dg) \\ &\leq \int_{B_n^c} U^2(g) \varXi(dg) \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

where $B_n = \{g \in L_2^{\uparrow}(\xi) : \ \|g\|_2 \leq n\}$, it is enough to show that $U\varphi_n(\|\cdot\|_2^2)$ can be approximated by functions from $\mathcal{FC}$.

Let U_n be the restriction of U on the ball B_n. Since $U_n \in L_2(B_n, \Xi|_{B_n})$ and the restriction $\Xi|_{B_n}$ of Ξ on B_n is a finite measure, the function U_n can be approximated in $L_2(B_n, \Xi|_{B_n})$ by functions of the form $u(\langle\cdot, h_1\rangle, \ldots, \langle\cdot, h_m\rangle)$, where $u \in C_b^\infty(\mathbb{R}^m)$ and $h_j \in L_2(\xi)$, $j \in [m]$, by the monotone class theorem (see, e.g. A0.6 [52]). Thus, for a fixed $\varepsilon > 0$ there exists a function $\widetilde{U} = u(\langle\cdot, h_1\rangle, \ldots, \langle\cdot, h_m\rangle)$ such that

$$\int_{B_n} \left(U_n(g) - \widetilde{U}(g)\right)^2 \Xi(dg) < \varepsilon.$$

Consequently,

$$\begin{aligned}
&\int_{L_2^{\uparrow}(\xi)} \left(U(g)\varphi_n\left(\|g\|_2^2\right) - \widetilde{U}(g)\varphi_n\left(\|g\|_2^2\right)\right)^2 \Xi(dg) \\
&\qquad = \int_{B_n} \left(U(g) - \widetilde{U}(g)\right)^2 \varphi_n^2\left(\|g\|_2^2\right) \Xi(dg) \\
&\qquad \leq \int_{B_n} \left(U_n(g) - \widetilde{U}(g)\right)^2 \Xi(dg) < \varepsilon.
\end{aligned}$$

It proves the proposition. $\square$

5.2 Differential operator and integration by parts formula

In this section we define the differential operator D on $\mathcal{FC}$.

For each $U \in \mathcal{FC}$ given by (12), i.e. $U = u(\langle\cdot, h_1\rangle, \ldots, \langle\cdot, h_m\rangle)\varphi(\|\cdot\|_2^2)$, the *differential operator* is defined as follows

$$\mathrm{D}U(g) := \mathrm{pr}_g\left[\nabla^{L_2}U(g)\right] = \varphi(\|g\|_2^2)\sum_{j=1}^m \partial_j u(\langle g, \mathbf{h}\rangle)\, \mathrm{pr}_g\, h_j + u(\langle g, \mathbf{h}\rangle)\varphi'(\|g\|_2^2)2g, \tag{13}$$

where ∇^{L_2} denotes the Fréchet derivative on L_2 and $\partial_j u(y) = \frac{\partial}{\partial y_j}u(y)$, $y \in \mathbb{R}^m$. For any function $U \in \mathcal{FC}$, DU is define by linearity.

A simple calculation gives the following statement.

Lemma 7 *For all $(q, x) \in Q^n \times E_0^n$, $n \geq 2$,*

$$\mathrm{D}U(\chi_n(q, x)) = \sum_{i=1}^n \frac{\partial}{\partial x_i} U(\chi_n(q, x)) \frac{\mathbb{I}_{[q_{i-1}, q_i)}}{(q_i - q_{i-1})}$$

and

$$\mathrm{D}U(\chi_1(x)) = \frac{d}{dx}U(\chi_1(x))\mathbb{I}_{[0,1]}.$$

In particular, for each $i \in [n]$

$$\langle \mathrm{D}U(\chi_n(q, x)), \mathbb{I}_{[q_{i-1}, q_i)}\rangle = \langle \nabla^{L_2}U(\chi_n(q, x)), \mathbb{I}_{[q_{i-1}, q_i)}\rangle = \frac{\partial}{\partial x_i}U(\chi_n(q, x)).$$

The definition of the differential operator and Lemma 7 immediately implies some trivial properties of D.

Remark 12 (i) For each $U \in \mathcal{FC}$, $\mathrm{D}U$ maps $L_2^{\uparrow}(\xi)$ into $L_2(\xi)$ and, in general, $\mathrm{D}U$ is not continuous, since $\mathrm{pr}_{\cdot} h$ is not, for each non constant function $h \in L_2(\xi)$.
(ii) D is a linear operator satisfying the Leibniz rule.
(iii) For each $U \in \mathcal{FC}$, $f \in L_2(\xi)$ and $g \in L_2^{\uparrow}(\xi)$

$$\mathrm{D}_f U(g) := \langle \mathrm{D}U(g), f\rangle = \lim_{\varepsilon \downarrow 0} \frac{U\left(g + \varepsilon \,\mathrm{pr}_g f\right) - U(g)}{\varepsilon}.$$

Now we prove the integration by parts formula. For this we first define the second order differential operator on $\mathcal{FC}$ in a similar way as in the finite dimensional case. We set for $U \in \mathcal{FC}$

$$L_0 U(g) = \begin{cases} \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} U(\chi_n(q,x)) \frac{1}{(q_i - q_{i-1})}, & g = \chi_n(q,x), \quad n \geq 2, \\ & (q,x) \in Q^n \times E_0^n, \\ \frac{d^2}{dx^2} U(\chi_1(x)), & x \in \mathbb{R}, \\ 0, & \text{otherwise.} \end{cases}$$

Using a simple calculation and Remark 4, we can prove the following lemma.

Lemma 8 *If $U \in \mathcal{FC}$ is given by (12), then*

$$\begin{aligned} L_0 U(g) = \varphi(\|g\|_2^2) &\sum_{i,j=1}^m \partial_i \partial_j u(\langle g, \mathbf{h}\rangle) \langle \mathrm{pr}_g h_i, \mathrm{pr}_g h_j\rangle \\ &+ u(\langle g, \mathbf{h}\rangle)\left[4\varphi''(\|g\|_2^2)\|g\|_2^2 + 2\varphi'(\|g\|_2^2)\cdot \sharp g\right] \\ &+ 2\sum_{j=1}^m \partial_j u(\langle g, \mathbf{h}\rangle)\varphi'(\|g\|_2^2)\langle \mathrm{pr}_g h_i, g\rangle, \quad g \in \mathcal{S}^{\uparrow}, \end{aligned}$$

and

$$L_0 U(g) = 0, \quad g \in L_2^{\uparrow}(\xi) \setminus \mathcal{S}^{\uparrow}.$$

Theorem 3 (Integration by parts formula) *Let $U, V \in \mathcal{FC}$. Then*

$$\begin{aligned} \int_{L_2^{\uparrow}(\xi)} \langle \mathrm{D}U(g), \mathrm{D}V(g)\rangle \Xi(dg) = &- \int_{L_2^{\uparrow}(\xi)} L_0 U(g) V(g) \Xi(dg) \\ &- \int_{L_2^{\uparrow}(\xi)} V(g) \langle \nabla^{L_2} U(g) - \mathrm{D}U(g), \xi\rangle \Xi(dg). \end{aligned}$$

In particular, if U is given by (12), then

$$\begin{aligned} \int_{L_2^{\uparrow}(\xi)} \langle \mathrm{D}U(g), \mathrm{D}V(g)\rangle \Xi(dg) = &- \int_{L_2^{\uparrow}(\xi)} L_0 U(g) V(g) \Xi(dg) \\ &- \int_{L_2^{\uparrow}(\xi)} \varphi(\|g\|_2^2) V(g) \sum_{j=1}^m \partial_j u(\langle g, \mathbf{h}\rangle) \langle h_j, \xi - \mathrm{pr}_g \xi\rangle \Xi(dg). \end{aligned} \tag{14}$$

We remark that $\nabla^{L_2}U(g)-\mathrm{D}U(g)$ coincides with the projection of $\nabla^{L_2}U(g)$ on the orthogonal complement of $L_2(g)$ in L_2.

Proof (Proof of Theorem 3) To prove the proposition, we will use Lemma 7 and the integration by parts formula for the Riemann integral.

So, first we note that

$$\int_{L_2^{\uparrow}(\xi)} \langle \mathrm{D}U(g), \mathrm{D}V(g)\rangle \Xi_1(dg) = -\int_{L_2^{\uparrow}(\xi)} L_0U(g)V(g)\Xi_1(dg). \tag{15}$$

Indeed, by (5) and Remark 11 (iii),

$$\begin{aligned}\int_{L_2^{\uparrow}(\xi)} \langle \mathrm{D}U(g), \mathrm{D}V(g)\rangle \Xi_1(dg) &= \int_{\mathbb{R}} \langle \mathrm{D}U(\chi_1(x)), \mathrm{D}V(\chi_1(x))\rangle dx \\ &= \int_{\mathbb{R}} \frac{d}{dx}U(\chi_1(x))\frac{d}{dx}V(\chi_1(x)))dx \\ &= -\int_{\mathbb{R}} \left(\frac{d^2}{dx^2}U(\chi_1(x))\right) V(\chi_1(x)))dx \\ &= -\int_{L_2^{\uparrow}(\xi)} L_0U(g)V(g)\Xi_1(dg).\end{aligned}$$

Next, we check that for each $n \geq 2$

$$\begin{aligned}\int_{L_2^{\uparrow}(\xi)} \langle \mathrm{D}U(g), \mathrm{D}V(g)\rangle \Xi_n(dg) = &-\int_{L_2^{\uparrow}(\xi)} L_0U(g)V(g)\Xi_n(dg) \\ &-\int_{L_2^{\uparrow}(\xi)} \langle \nabla^{L_2}U(g) - \mathrm{D}U(g), \xi\rangle V(g)\Xi_{n-1}(dg).\end{aligned} \tag{16}$$

To show this, we reduce the integral over Ξ_n to the Riemann-Stieltjes integral similarly as in the previous case. So, by Lemma 2 (i), we have

$$\begin{aligned}&\int_{L_2^{\uparrow}(\xi)} \langle \mathrm{D}U(g), \mathrm{D}V(g)\rangle \Xi_n(dg) \\ &= \int_{Q^n} \prod_{i=1}^{n}(q_i - q_{i-1}) \left[\int_{E^n} \langle \mathrm{D}U(\chi(q,x)), \mathrm{D}V(\chi(q,x))\rangle \lambda_n(dx)\right] d\xi^{\otimes(n-1)}(q).\end{aligned}$$

Next, we fix $q \in Q^n$ and apply to the integral over λ_n the usual integration by parts formula. Consequently, using Lemma 7, we obtain

$$\begin{aligned}
&\int_{E^n} \langle \mathrm{D}U(\chi(q,x)), \mathrm{D}V(\chi(q,x))\rangle \lambda_n(dx) \\
&\quad = \int_{E^n} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} U(\chi(q,x)) \frac{\partial}{\partial x_i} V(\chi(q,x)) \frac{1}{q_i - q_{i-1}} \lambda_n(dx) \\
&\quad = -\int_{E^n} \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} U(\chi(q,x)) \right) \frac{1}{q_i - q_{i-1}} V(\chi(q,x)) \lambda_n(dx) \\
&\quad + \sum_{i=1}^{n} \int_{E^{n-1}} \left[\left(\frac{\partial}{\partial x_i} U(\chi(q,x)) \right) V(\chi(q,x)) \right] \Big|_{x_i = x_{i-1}}^{x_i = x_{i+1}} \frac{\lambda_{n-1}(dx^{(i)})}{q_i - q_{i-1}} \\
&\quad =: I_1(q) + I_2(q),
\end{aligned}$$

where $x^{(i)} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, $x_0 = -\infty$ and $x_{n+1} = +\infty$.

By the definition of the operator L_0 and Lemma 2 (i), we have that

$$\int_{Q^n} \left(\prod_{i=1}^{n} (q_i - q_{i-1}) \right) I_1(q) d\xi^{\otimes(n-1)}(q) = -\int_{L_2^{\uparrow}(\xi)} L_0 U(g) V(g) \varXi_n(dg).$$

Next we rewrite $I_2(q)$. By Lemma 7, we obtain

$$\begin{aligned}
I_2(q) &= \sum_{i=1}^{n} \int_{E^{n-1}} \left[\left\langle \nabla^{L_2} U(\chi(q,x)), \mathbb{I}_{[q_{i-1}, q_i)} \right\rangle V(\chi(q,x)) \right] \Big|_{x_i = x_{i-1}}^{x_i = x_{i+1}} \frac{\lambda_{n-1}(dx^{(i)})}{q_i - q_{i-1}} \\
&= \sum_{i=1}^{n-1} \int_{E^{n-1}} \left\langle \nabla^{L_2} U(\chi(q^{(i)}, x)), e_i(q) - e_{i+1}(q) \right\rangle V(\chi(q^{(i)}, x)) \lambda_{n-1}(dx),
\end{aligned}$$

where $q^{(i)}$ is defined similarly as $x^{(i)}$, removing the i-th coordinate, and $e_i(q) := \frac{\mathbb{I}_{[q_{i-1}, q_i)}}{q_i - q_{i-1}}$, $i \in [n]$. For a simplification of notation, we denote

$$c_n(q) = \prod_{i=1}^{n} (q_i - q_{i-1}).$$

Then

$$\begin{aligned}
\int_{Q^n} c_n(q) I_2(q) d\xi^{\otimes(n-1)}(q) &= \sum_{i=1}^{n} \int_{E^{n-1}} \left[\int_{Q^n} c_n(q) \Big\langle \nabla^{L_2} U(\chi(q^{(i)}, x)), e_i(q) \right. \\
&\quad \left. - e_{i+1}(q) \Big\rangle V(\chi(q^{(i)}, x)) d\xi^{\otimes(n-1)}(q) \right] \lambda_{n-1}(dx) \\
&= \sum_{i=1}^{n} \int_{E^{n-1}} \left[\int_{Q^{n-1}} c_{n-1}(q^{(i)}) \left\langle \nabla^{L_2} U(\chi(q^{(i)}, x)), f(q^{(i)}) \right\rangle \right. \\
&\quad \left. \cdot V(\chi(q^{(i)}, x)) d\xi^{\otimes(n-2)}(q^{(i)}) \right] \lambda_{n-1}(dx),
\end{aligned}$$

where

$$f(q^{(i)}) := \int_{q_{i-1}}^{q_{i+1}} \frac{(q_{i+1}-q_i)(q_i-q_{i-1})}{q_{i+1}-q_{i-1}}(e_i(q)-e_{i+1}(q))d\xi(q_i).$$

Integrating by parts, we obtain

$$\begin{aligned} f(q^{(i)})(r) &= \left(\int_r^{q_{i+1}} \frac{q_{i+1}-q_i}{q_{i+1}-q_{i-1}}d\xi(q_i) - \int_{q_{i-1}}^r \frac{q_i-q_{i-1}}{q_{i+1}-q_{i-1}}d\xi(q_i)\right)\mathbb{I}_{[q_{i-1},q_{i+1})}(r) \\ &= \left(\frac{1}{q_{i+1}-q_{i-1}}\left\langle \xi, \mathbb{I}_{[q_{i-1},q_{i+1})}\right\rangle - \xi(r)\right)\mathbb{I}_{[q_{i-1},q_{i+1})}(r), \quad r\in[0,1]. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{Q^n} c_n(q)I_2(q)d\xi^{\otimes(n-1)}(q) = \int_{Q^{n-1}} c_{n-1}(q)\bigg[\int_{E^{n-1}} &\left\langle \nabla^{L_2}U(\chi(q,x)), \mathrm{pr}_{\chi(q,\widetilde{x})}\,\xi - \xi\right\rangle \\ &\cdot V(\chi(q,x))\lambda_{n-1}(dx)\bigg]d\xi^{\otimes(n-2)}(q), \end{aligned}$$

where $\widetilde{x}$ is any point from E_0^{n-1} (here $\mathrm{pr}_{\chi(q,\widetilde{x})} = \mathrm{pr}_{\chi(q,\widetilde{y})}$ for all $\widetilde{x},\widetilde{y}\in E_0^{n-1}$). This immediately implies

$$\begin{aligned} &\int_{Q^n}\left(\prod_{i=1}^n(q_i-q_{i-1})\right)I_2(q)d\xi^{\otimes(n-1)}(q) \\ &= -\int_{L_2^{\uparrow}(\xi)}\langle\nabla^{L_2}U(g), \xi-\mathrm{pr}_g\,\xi\rangle V(g)\varXi_{n-1}(dg) \\ &= -\int_{L_2^{\uparrow}(\xi)}\langle\nabla^{L_2}U(g)-\mathrm{D}U(g), \xi\rangle V(g)\varXi_{n-1}(dg), \end{aligned}$$

where we have used the trivial equality

$$\langle\nabla^{L_2}U(g)-\mathrm{D}U(g),\xi\rangle = \langle\nabla^{L_2}U(g),\xi-\mathrm{pr}_g\,\xi\rangle \tag{17}$$

It proves (16).

Next, summing (15) and (16) over n and using Remark 9, we obtain the integration by parts formula. (14) easily follows from (17) and the equality $\langle g, \mathrm{pr}_g\,\xi-\xi\rangle = 0$. The theorem is proved. □

The same argument as in the proof of the previous theorem gives the adjoint operator for $\mathrm{D}_f = \langle\mathrm{D}\cdot, f\rangle$, $f\in L_2^{\uparrow}(\xi)$.

Proposition 4 *For each $U,V\in\mathcal{FC}$ and $f\in L_2$*

$$\begin{aligned} \int_{L_2^{\uparrow}(\xi)}(\mathrm{D}_fU(g))\,V(g)\varXi(dg) = -&\int_{L_2^{\uparrow}(\xi)} U(g)\mathrm{D}_fV(g)\varXi(dg) \\ &- \int_{L_2^{\uparrow}(\xi)} U(g)V(g)\langle f,\xi-\mathrm{pr}_g\,\xi\rangle\varXi(dg). \end{aligned}$$

Remark 13 The adjoint operator for D_f is given by the formula

$$\mathrm{D}_f^*U(g) = -\mathrm{D}_fU(g) - \langle f,\xi-\mathrm{pr}_g\,\xi\rangle U(g), \quad g\in L_2^{\uparrow}(\xi),\ \ U\in\mathcal{FC}.$$

5.3 The Dirichlet form $(\mathcal{E}, \mathbb{D})$

We define

$$\mathcal{E}(U,V) = \frac{1}{2}\int_{L_2^{\uparrow}(\xi)} \langle \mathrm{D}U(g), \mathrm{D}V(g)\rangle \Xi(dg), \quad U, V \in \mathcal{FC}.$$

Then $(\mathcal{E}, \mathcal{FC})$ is a densely defined positive definite symmetric bilinear form on $L_2(L_2^{\uparrow}(\xi), \Xi)$, by Proposition 3. The integration by parts formula implies that there exists a negative definite symmetric linear operator L on $L_2(\Xi)$, given by

$$\begin{aligned} LU(g) :&= \frac{1}{2}\left[L_0 U(g) + \langle \nabla^{L_2} U(g) - \mathrm{D}U(g), \xi\rangle\right] \\ &= \frac{1}{2}\left[L_0 U(g) + \varphi(\|g\|_2^2)\sum_{j=1}^{m} \partial_j u(\langle g, \mathbf{h}\rangle)\langle \xi - \mathrm{pr}_g\, \xi, h_j\rangle\right], \quad g \in L_2^{\uparrow}(\xi), \end{aligned} \tag{18}$$

if $U \in \mathcal{FC}$ is defined by (12), such that

$$\mathcal{E}(U,V) = -\langle LU, V\rangle_{L_2(\Xi)}.$$

Consequently, by Proposition I.3.3 [39], $(\mathcal{E}, \mathcal{FC})$ is closable on $L_2(\Xi)$.

Definition 1 The closure $(\mathcal{E}, \mathcal{FC})$ on $L_2(\Xi)$ is denoted by $(\mathcal{E}, \mathbb{D})$.

Remark 14 We can extend the differential operator D to $\mathbb{D}$, letting

$$\mathrm{D}U := \lim_{n\to\infty} \mathrm{D}U_n \quad \text{in} \quad L_2(\Xi),$$

if $\{U_n,\ n \geq 1\} \subset \mathcal{FC}$ converges to $U \in \mathbb{D}$ with respect to the norm $\mathcal{E}_1^{\frac{1}{2}}$, where $\mathcal{E}_1 := \mathcal{E}(\cdot,\cdot) + \langle \cdot,\cdot\rangle_{L_2(\Xi)}$. Then, for all $U, V \in \mathbb{D}$

$$\mathcal{E}(U,V) = \frac{1}{2}\int_{L_2^{\uparrow}(\xi)} \langle \mathrm{D}U(g), \mathrm{D}V(g)\rangle \Xi(dg). \tag{19}$$

Next we are going to check that $(\mathcal{E}, \mathbb{D})$ is a Dirichlet form. For this we need an analog of the chain rule.

Lemma 9 *Let $F \in C^1(\mathbb{R}^k)$, $F(0) = 0$ and $U_j \in \mathcal{FC}$, $j \in [k]$. Then the composition $F(U) = F(U_1, \ldots, U_k)$ belongs to $\mathbb{D}$ and*

$$\mathrm{D}F(U)(g) = \sum_{j=1}^{k} \partial_j F(U(g))\mathrm{D}U_j(g), \quad g \in L_2^{\uparrow}(\xi).$$

Proof We will prove the lemma, using the approximation of F by the Bernstein polynomials and the fact that $\mathcal{FC}$ is an associative algebra (see Remark 11 (i)).

Since U_j, $j \in [k]$, belong to $\mathcal{FC}$, they are bounded by a constant M, i.e. $|U_j(g)| \leq M$ for all $g \in L_2^{\uparrow}(\xi)$. Next let polynomials $P_n^M(F;\cdot)$, $n \geq 1$, be defined by (39). Then by Lemma 16,

$$\left|P_n^M(F;U(g)) - F(U(g))\right| \leq \sup_{x\in[-M,M]^k} \left|P_n^M(F;x) - F(x)\right| \mathbb{I}_{\operatorname{supp} U}(g) \to 0,$$

as $n \to \infty$, where $\operatorname{supp} U := \bigcup_{j=1}^k \operatorname{supp} U_j$. Hence, by remarks 9, 11 (ii) and the dominated convergence theorem, we have that $\{P_n^M(F;U)\}_{n\geq 1}$ converges to $F(U)$ in $L_2(\Xi)$.

Remark 11 (ii) and the fact that $P_n^M(F;0) = 0$ imply that $P_n^M(F;U) \in \mathcal{FC}$. Moreover, the Leibniz rule for D (see Remark 12) yields

$$\mathrm{D}P_n^M(F;U)(g) = \sum_{j=1}^k \partial_j P_n^M(F;U(g))\mathrm{D}U_j(g), \quad g \in L_2^{\uparrow}(\xi).$$

Using the estimate

$$\begin{aligned}&\left|\partial_j P_n^M(F;U(g))\mathrm{D}U_j(g) - \partial_j F(U(g))\mathrm{D}U_j(g)\right| \\ &\leq \sup_{x\in[-M,M]^k} \left|\partial_j P_n^M(F;x) - \partial_j F(x)\right| |\mathrm{D}U_j(g)|,\end{aligned}$$

Lemma 16 and the dominated convergence theorem, we obtain that $\{\mathrm{D}P_n^M(F;U)\}_{n\geq 1}$ converges to $\sum_{j=1}^k \partial_j F(U)\mathrm{D}U_j$ in $L_2(\Xi)$. It finishes the proof of the lemma. □

Corollary 3 *For each $u \in C_b^1(\mathbb{R}^m)$, $h_j \in L_2(\xi)$, $j \in [m]$, and $\varphi \in C_0^\infty(\mathbb{R})$ the function $U = u(\langle\cdot, h_1\rangle, \ldots, \langle\cdot, h_m\rangle)\varphi(\|\cdot\|_2^2)$, belongs to $\mathbb{D}$ and $\mathrm{D}U$ is given by formula (13).*

Proof Let $\psi \in C_0^\infty(\mathbb{R})$ and $\psi = 1$ on $\operatorname{supp}\varphi$. We set

$$V_j = \langle\cdot, h_j\rangle\psi(\|\cdot\|_2^2), \quad j \in [m],$$

and

$$V_{m+1} = \varphi(\|\cdot\|_2^2).$$

It is easy to see that $V_j \in \mathcal{FC}$ for all $j \in [m+1]$ (since $\langle\cdot, h_j\rangle$ can be replaced by $u(\langle\cdot, h_j\rangle)$ in the definition of V_j for some $u \in C_b^\infty(\mathbb{R})$). Then, by Lemma 9,

$$U = F(V_1, \ldots, V_m, V_{m+1})$$

belongs to $\mathbb{D}$, where $F(x_1 \ldots, x_m, x_{m+1}) = u(x_1, \ldots, x_m) \cdot x_{m+1}$. Moreover, a simple calculation gives that $\mathrm{D}U = \mathrm{D}F(V)$ is given by (13). It proves the corollary. □

Next we give the analog of the chain rule for D that easily follows from Lemma 9 and the closability of $(\mathcal{E}, \mathbb{D})$.

Proposition 5 *Let $F \in C_b^1(\mathbb{R}^k)$, $F(0) = 0$ and $U_j \in \mathbb{D}$, $j \in [k]$. Then the composition $F(U) = F(U_1, \ldots, U_k)$ belongs to $\mathbb{D}$ and*

$$\mathrm{D}F(U)(g) = \sum_{j=1}^{k} \partial_j F(U(g))\mathrm{D}U_j(g), \quad g \in L_2^{\uparrow}(\xi).$$

Now we are ready to prove that $(\mathcal{E}, \mathbb{D})$ is a Dirichlet form on $L_2(L_2^{\uparrow}(\xi), \varXi)$. For $U, V : L_2^{\uparrow}(\xi) \to \mathbb{R}$ we set

$$U \wedge V = \min\{U, V\} \quad \text{and} \quad U \vee V = \max\{U, V\}.$$

Proposition 6 *The bilinear form $(\mathcal{E}, \mathbb{D})$ is a symmetric Dirichlet form on $L_2(L_2^{\uparrow}(\xi), \varXi)$, that is, for all $U \in \mathbb{D}$ the function $(U \vee 0) \wedge 1$ belongs to $\mathbb{D}$ and*

$$\mathcal{E}((U \vee 0) \wedge 1, (U \vee 0) \wedge 1) \leq \mathcal{E}(U, U).$$

Proof To prove the proposition, we need to show that for each $U \in \mathbb{D}$ and $\varepsilon > 0$ there exists a function $F_\varepsilon : \mathbb{R} \to [-\varepsilon, 1 + \varepsilon]$ such that $F_\varepsilon(x) = x$ for all $x \in [0, 1]$, $0 \leq F_\varepsilon(x_2) - F_\varepsilon(x_1) \leq x_2 - x_1$ if $x_1 \leq x_2$, $F_\varepsilon(U) \in \mathbb{D}$ and

$$\limsup_{\varepsilon \to 0} \mathcal{E}(F_\varepsilon(U), F_\varepsilon(U)) \leq \mathcal{E}(U, U),$$

according to Proposition I.4.7 [39].

We take for $\varepsilon > 0$ an arbitrary non decreasing continuously differentiable function $F_\varepsilon : \mathbb{R} \to [-\varepsilon, 1 + \varepsilon]$ such that $|F'(x)| \leq 1$, $x \in \mathbb{R}$, and $F_\varepsilon(x) = x$ for all $x \in [0, 1]$. Then it is clear that $0 \leq F_\varepsilon(x_2) - F_\varepsilon(x_1) \leq x_2 - x_1$ if $x_1 \leq x_2$. By Proposition 5, $F_\varepsilon(U) \in \mathbb{D}$ and

$$\limsup_{\varepsilon \to 0} \mathcal{E}(F_\varepsilon(U), F_\varepsilon(U)) = \frac{1}{2} \limsup_{\varepsilon \to 0} \int_{L_2^{\uparrow}(\xi)} |F_\varepsilon'(U(g))|^2 \|\mathrm{D}U(g)\|_2^2 \varXi(dg) \leq \mathcal{E}(U, U).$$

It proves the proposition. □

Lemma 10 *Let U, V in $\mathbb{D}$. Then $U \vee V \in \mathbb{D}$ and*

$$\mathcal{E}(U \vee V, U \vee V) \leq \mathcal{E}(U, U) \vee \mathcal{E}(V, V). \tag{20}$$

Proof The fact that $U \vee V \in \mathbb{D}$ follows from Proposition I.4.11 [39]. Inequality (20) can be proved similarly as Lemma IV.4.1 [39]. □

Lemma 11 *Let $U, V \in \mathbb{D}$ and $|U| \vee \|\mathrm{D}U\|_2$ is bounded $\varXi$-a.e. Then $U \cdot V \in \mathbb{D}$ and $\mathrm{D}(U \cdot V) = (\mathrm{D}U) \cdot V + U \cdot \mathrm{D}V$.*

Proof The lemma follows from Corollary I.4.15 and Proposition 5, using an approximation (w.r.t $\mathcal{E}^{\frac{1}{2}}$-norm) of V by bounded functions. □

6 Quasi-regularity of the Dirichlet form $(\mathcal{E}, \mathbb{D})$

In this section we prove that $(\mathcal{E}, \mathbb{D})$ is quasi-regular that will imply the existence of a Markov process in $L_2^{\uparrow}(\xi)$ that is properly associated with $(\mathcal{E}, \mathbb{D})$.

6.1 Functions with compact support

The aim of this section is to prove that the domain $\mathbb{D}$ of the Dirichlet form contains a rich enough subset of functions with compact support.

Lemma 12 *For each $p \in \left[2, \frac{5}{2}\right]$, $g_0 \in L_2^{\uparrow}(\xi)$ and $\varphi \in C_0^{\infty}(\mathbb{R})$ the function $\varphi(\|\cdot -g_0\|_p^p)$ belongs to $\mathbb{D}$. Moreover, $\mathrm{D}\varphi(\|\cdot -g_0\|_2^2)(g) = 2\varphi'(\|g-g_0\|_2^2)\operatorname{pr}_g(g-g_0)$ for all $g \in L_2(\Xi)$.*

Proof For simplicity we give the proof for $g_0 = 0$.

Let $\{h_n\}_{n\geq 1} \subseteq L_\infty$ be a dense subset in L_q with $\|h_n\|_q = 1$, where $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\|g\|_p = \sup_{n\geq 1} |\langle g, h_n\rangle| = \sup_{n\geq 1} \left| \int_0^1 g(s)h_n(s)ds \right|.$$

Next we take functions $\psi_1, \psi_2 \in C_0^{\infty}(\mathbb{R})$ such that $\psi_1 = 1$ on $[-M-1, M+1]$, $\operatorname{supp}\psi_1 \subseteq [-2M-2, 2M+2]$, $\psi_2 = 1$ on $[-M, M]$ and $\operatorname{supp}\psi_2 \subseteq [-M-1, M+1]$, where M is chosen such that the interval $[-M^{\frac{p}{2}}, M^{\frac{p}{2}}]$ contains $\operatorname{supp}\varphi$, and define for each $n \geq 1$

$$U_n(g) := \max_{i\in[n]} |\langle g, h_i\rangle|^p \psi_1(\|g\|_2^2), \quad g \in L_2^{\uparrow}(\xi),$$

and

$$V_n(g) := \varphi(U_n(g))\psi_2(\|g\|_2^2) = \varphi\left(\max_{i\in[n]} |\langle g, h_i\rangle|^p\right)\psi_2(\|g\|_2^2), \quad g \in L_2^{\uparrow}(\xi).$$

Let us note that $U_n \in \mathbb{D}$, $n \geq 1$, by Corollary 3 and Lemma 10. Hence, due to Proposition 5, V_n also belongs to $\mathbb{D}$ for all $n \geq 1$.

By the choice of the function ψ_2, it is easy to see that for all $g \in L_p^{\uparrow}$

$$V_n(g) \to \varphi(\|g\|_p^p)\psi_2(\|g\|_2^2) = \varphi(\|g\|_p^p), \quad \text{as} \;\; n \to \infty,$$

and, consequently, $\{V_n\}_{n\geq 1}$ converges to $\varphi(\|\cdot\|_p^p)$ Ξ-a.e., by Corollary 2. Moreover,

$$|V_n(g) - \varphi(\|g\|_p^p)| \leq 2\|\varphi\|_\infty \mathbb{I}_{\{\|g\|_2^2 \leq M+1\}}, \quad n \geq 1.$$

The dominated convergence theorem implies that $\{V_n\}_{n\geq 1}$ converges to $\varphi(\|\cdot\|_p^p)$ in $L_2(\Xi)$.

Next, using Proposition 5 and Lemma 10, we can estimate

$$\begin{aligned}
\mathcal{E}(V_n, V_n) &\leq \frac{1}{2}\|\varphi'\|_\infty^2 \|\psi_2\|_\infty^2 \int_{L_2^\uparrow(\xi)} \|\mathrm{D}U_n\|_2^2 \Xi(dg) \\
&+ 2\|\varphi\|_\infty^2 \int_{L_2^\uparrow(\xi)} \left(\psi_2'(\|g\|_2^2)\right)^2 \|g\|_2^2 \Xi(dg) \\
&\leq \frac{1}{2}\|\varphi'\|_\infty^2 \|\psi_2\|_\infty^2 \max_{i\in[n]} \int_{L_2^\uparrow(\xi)} \Big[\psi_1^2(\|g\|_2^2) p^2 |\langle g, h_i\rangle|^{2p-2} \|\operatorname{pr}_g h_i\|_2^2 \\
&+ 4|\langle g, h_i\rangle|^p \left(\psi_1'(\|g\|_2^2)\right)^2 \|g\|_2^2\Big] \Xi(dg) \\
&+ 2\|\varphi\|_\infty^2 \|\psi_2'\|_\infty^2 \int_{L_2^\uparrow(\xi)} \|g\|_2^2 \mathbb{I}_{\{\|g\|_2^2 \leq M+1\}} \Xi(dg) \\
&\leq \frac{1}{2} p^2 \|\varphi'\|_\infty^2 \|\psi_2\|_\infty^2 \|\psi_1\|_\infty^2 \\
&\cdot \max_{i\in[n]} \int_{L_2^\uparrow(\xi)} |\langle g, h_i\rangle|^{2p-2} \|\operatorname{pr}_g h_i\|_2^2 \mathbb{I}_{\{\|g\|_2^2 \leq M+1\}} \Xi(dg) \\
&+ 2\|\varphi'\|_\infty^2 \|\psi_2\|_\infty^2 \|\psi_1\|_\infty^2 \int_{L_2^\uparrow(\xi)} |\langle g, h_i\rangle|^p \|g\|_2^2 \mathbb{I}_{\{\|g\|_2^2 \leq 2M+2\}} \Xi(dg) \\
&+ 2\|\varphi\|_\infty^2 \|\psi_2'\|_\infty^2 \int_{L_2^\uparrow(\xi)} \|g\|_2^2 \mathbb{I}_{\{\|g\|_2^2 \leq M+1\}} \Xi(dg).
\end{aligned}$$

Using Hölder's inequality $|\langle g, h_i\rangle| \leq \|h_i\|_q \|g\|_p = \|g\|_p$ and Lemma 3, we have that

$$\sup_{n\in\mathbb{N}} \mathcal{E}(V_n, V_n) < \infty,$$

if $p \in \left[2, \frac{5}{2}\right]$.

Hence, Lemma I.2.12 [39] yields $\varphi(\|\cdot\|_p^p) \in \mathbb{D}$ and

$$\mathcal{E}(\varphi(\|\cdot\|_p^p), \varphi(\|\cdot\|_p^p)) \leq \liminf_{n\to\infty} \mathcal{E}(V_n, V_n). \tag{21}$$

In order to compute $\mathrm{D}\varphi(\|\cdot - g_0\|_2^2)$, we take an orthonormal basis $\{h_n\}_{n\geq 1}$ in L_2 and note that

$$\|g - g_0\|^2 = \sum_{n=1}^{\infty} (\langle g, h_n\rangle - \langle g_0, h_n\rangle)^2.$$

Taking $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi = 1$ on an interval $[-M, M]$ that contains $\operatorname{supp}\varphi$ and setting

$$W_n(g) = \varphi\left(\sum_{i=1}^{n} (\langle g, h_i\rangle - \langle g_0, h_i\rangle)^2\right) \psi(\|g\|_2^2), \quad g \in L_2^\uparrow(\xi),$$

a simple calculation shows that

$$W_n \to \varphi(\|\cdot - g_0\|_2^2)$$

and

$$\|\mathrm{D}W_n - \mathrm{D}\varphi(\|\cdot - g_0\|_2^2)\|_2 \to 0$$

in $L_2(\Xi)$ as $n \to \infty$. The lemma is proved. □

Corollary 4 *For each $\varphi \in C_0^\infty(\mathbb{R})$ and $g_0 \in L_2^\uparrow(\xi)$ the function $U = \|\cdot - g_0\|_2 \varphi(\|\cdot\|_2^2)$ belongs to $\mathbb{D}$. Moreover, $\|DU\| \le 1$ Ξ-a.e. on $B_r = \{g \in L_2^\uparrow(\xi) : \|g\|_2 \le r\}$, if $\varphi = 1$ on $[-r^2, r^2]$.*

Proof We take $\psi \in C_0^\infty(\mathbb{R})$ such that $\psi = 1$ on an interval $[-M, M]$ which contains $\operatorname{supp}\varphi$. For each $\delta > 0$, we set

$$V_\delta(g) = \left(\|g - g_0\|_2^2 \vee \delta^2\right) \psi(\|g\|_2^2), \quad g \in L_2^\uparrow(\xi).$$

Let $\psi_\delta \in C_b^\infty(\mathbb{R})$ and $\psi_\delta(x) = \sqrt{|x|}$ for all $\delta \le |x| \le \sup_g |V_\delta(g)|$. Then by lemmas 10, 12 and Proposition 5, the function $U_\delta = \psi_\delta(V_\delta)\varphi(\|\cdot\|_2^2)$ belongs to $\mathbb{D}$ and

$$\mathcal{E}(U_\delta, U_\delta) \le C < \infty$$

for all $\delta > 0$. Since $U_\delta \to U = \|\cdot - g_0\|_2 \varphi(\|\cdot\|_2^2)$ in $L_2(\Xi)$ as $\delta \to 0$, the function U belongs to $\mathbb{D}$, by Lemma I.2.12 [39].

A simple calculation shows that $\|\mathrm{D}U_\delta\| \le 1$ Ξ-a.e. on B_r (if $\varphi = 1$ on $[-r^2, r^2]$). Hence, by Lemma I.2.12 [39], $\|\mathrm{D}U\| \le 1$ Ξ-a.e. on B_r. □

Let $\mathcal{FC}_0$ be the linear span of the set of functions on $L_2^\uparrow(\xi)$ which have a form

$$U = u(\langle\cdot, h_1\rangle, \dots, \langle\cdot, h_m\rangle)\varphi(\|\cdot\|_p^p) = u(\langle\cdot, \mathbf{h}\rangle)\varphi(\|\cdot\|_p^p), \tag{22}$$

where $p \in \left(2, \frac{5}{2}\right]$, $u \in C_b^\infty(\mathbb{R}^m)$, $\varphi \in C_0^\infty(\mathbb{R})$ and $h_j \in L_2(\xi)$, $j \in [m]$.

Remark 15 Each function from $\mathcal{FC}_0$ has a compact support in $L_2^\uparrow(\xi)$, by Lemma 5.1 [31].

Proposition 7 *The set $\mathcal{FC}_0$ is dense in $\mathbb{D}$ with respect to the norm $\mathcal{E}_1^{\frac{1}{2}}$.*

Proof First we note that by Proposition 5 and Lemma 12, $\mathcal{FC}_0 \subset \mathbb{D}$.

To prove the proposition, it is enough to show that each element of $\mathcal{FC}$ can be approximated by elements from $\mathcal{FC}_0$. So, let $U \in \mathcal{FC}$ is given by (12), i.e. $U = u(\langle\cdot, \mathbf{h}\rangle)\varphi(\|\cdot\|_2^2)$. By the dominated convergence theorem and Lemma 4,

$$U_p = u(\langle\cdot, \mathbf{h}\rangle)\varphi(\|\cdot\|_p^p) \to U \quad \text{in} \quad L_2(\Xi) \quad \text{as} \quad p \downarrow 2.$$

Next, using Proposition 5, we can estimate,

$$\begin{aligned}\mathcal{E}(U_p,U_p) &= \frac{1}{2}\int_{L_2^{\uparrow}(\xi)} \|\mathrm{D}U_p(g)\|_2^2 \Xi(dg)\\ &\le 2^{m-1}\sum_{j=1}^m \int_{L_2^{\uparrow}(\xi)} \varphi^2(\|g\|_p^p)(\partial_j u(\langle g,\mathbf{h}\rangle))^2 \|\operatorname{pr}_g h_j\|_2^2 \Xi(dg)\\ &+ 2^{m-1}\int_{L_2^{\uparrow}(\xi)} (u(\langle g,\mathbf{h}\rangle))^2 \|\mathrm{D}\varphi(\|\cdot\|_p^p)(g)\|_2^2 \Xi(dg)\\ &\le 2^{m-1}\|\varphi\|_\infty^2 \sum_{j=1}^m \|\partial_j u\|_\infty^2 \|h_j\|_2^2 \int_{L_2^{\uparrow}(\xi)} \varphi^2(\|g\|_p^p)\Xi(dg)\\ &+ \|u\|_\infty^2 \mathcal{E}(\varphi(\|\cdot\|_p^p),\varphi(\|\cdot\|_p^p)) < C\end{aligned}$$

uniformly in $p \in \left(2, \frac{5}{2}\right]$, by estimate (21), Lemma 3 and the inequality $\|g\|_2 \le \|g\|_p$ for $p > 2$.

Hence, by Lemma I.2.12 [39], there exists a subsequence $\{U_{p_k}\}_{k\ge 1}$ $(p_k \downarrow 2)$ such that its Cesaro mean

$$V_n = \frac{1}{n}\sum_{k=1}^n U_{n_k} \to U$$

in $\mathbb{D}$ (w.r.t. $\mathcal{E}_1^{\frac{1}{2}}$-norm) as $n \to \infty$. Since, $\mathcal{FC}_0$ is linear, $V_n \in \mathcal{FC}_0$, $n \in \mathbb{N}$. So, it gives the needed approximation. The proposition is proved. □

6.2 Quasi-regularity and local property of $(\mathcal{E},\mathbb{D})$

The aim of this section is to show that $(\mathcal{E},\mathbb{D})$ is a quasi-regular Dirichlet form. Let

$$\mathbb{D}_K = \left\{U \in \mathbb{D}:\ U = 0 \ \ \Xi\text{-a.e. on } L_2^{\uparrow}(\xi)\setminus K\right\}.$$

We recall that an increasing sequence $\{K_n\}_{n\ge 1}$ of closed subsets of $L_2^{\uparrow}(\xi)$ is called an *$\mathcal{E}$-nest*[2] if $\bigcup_{n=1}^\infty \mathbb{D}_{K_n}$ is dense in $\mathbb{D}$ (w.r.t. $\mathcal{E}^{\frac{1}{2}}$-norm).

Proposition 8 *The Dirichlet form $(\mathcal{E},\mathbb{D})$ is quasi-regular, that is, it has the following properties*

(i) there exists an $\mathcal{E}$-nest $\{K_n\}_{n\ge 1}$ consisting of compact sets;

(ii) there exists a dense subset of $\mathbb{D}$ (w.r.t. $\mathcal{E}_1^{\frac{1}{2}}$-norm) whose elements have $\mathcal{E}$-quasi-continuous Ξ-version;

(iii) there exists $U_n \in \mathbb{D}$, $n \in \mathbb{N}$, having $\mathcal{E}$-quasi-continuous Ξ-version $\widetilde{U}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $A \subset L_2^{\uparrow}(\xi)$ such that $\{\widetilde{U}_n,\ n \in \mathbb{N}\}$ separates points of $L_2^{\uparrow}(\xi)\setminus A$.

[2] The definitions of $\mathcal{E}$-nest, $\mathcal{E}$-quasi-continuity, quasi-regularity and local property are taken from [39] (see definitions III.2.1, III.3.2, IV.3.1 and V.1.1, respectively)

Proof Properties (ii) and (iii) follow from the fact that $\mathcal{FC}$ is dense in $\mathbb{D}$ (w.r.t. $\mathcal{E}^{\frac{1}{2}}$-norm) and $\mathcal{FC}$ separates points, since $\{\langle\cdot,h\rangle,\ h\in L_2\}$ separates the points of $L_2^{\uparrow}(\xi)$.

To prove (i), we set

$$K_n=\left\{g\in L_2^{\uparrow}(\xi):\ \|g\|_{2+\frac{1}{n}}\le n\right\}.$$

Then $\{K_n\}_{n\ge 1}$ is an increasing sequence of compact sets, by Lemma 5.1 [31]. Moreover, it is easily seen that

$$\mathcal{FC}_0\subseteq\bigcup_{n=1}^{\infty}\mathbb{D}_{K_n}.$$

Consequently, Proposition 7 yields (i). It proves the proposition. □

Proposition 9 *The Dirichlet form* $(\mathcal{E},\mathbb{D})$ *has the local property, that is,* $\mathcal{E}(U,V)=0$ *for all* $U,V\in\mathbb{D}$ *with* $\operatorname{supp}(U\cdot\varXi)\cap\operatorname{supp}(V\cdot\varXi)=\emptyset$ *and* $\operatorname{supp}(U\cdot\varXi)$, $\operatorname{supp}(V\cdot\varXi)$ *compact.*

Proof Let $U\in\mathbb{D}$ with $K_U:=\operatorname{supp}(U\cdot\varXi)$ compact. First we note that the equality $U=0$ $\varXi$-a.e. on a ball $B_r(g_0)=\{g\in L_2^{\uparrow}(\xi):\ \|g-g_0\|_2<r\}$ implies $\mathrm{D}U=0$ $\varXi$-a.e. on $B_r(g_0)$. Indeed, let $K_U\subset B_R(g_0)$ for some constant $R>0$. We take $\varepsilon\in(0,1)$ and $\varphi\in C_0^{\infty}(\mathbb{R})$ such that $\varphi(x)=0$ for all $|x|\le(1-\varepsilon)r^2$ and $\varphi(x)=1$ for all $r^2\le|x|\le R^2$. Then by lemmas 12 and 11, we can conclude that for all $g\in L_2^{\uparrow}(\xi)$

$$\begin{aligned}\mathrm{D}U(g)&=\mathrm{D}\left[U\varphi(\|\cdot-g_0\|_2^2)\right](g)\\&=(\mathrm{D}U(g))\varphi(\|g-g_0\|_2^2)+2U(g)\varphi'(\|g-g_0\|_2^2)g.\end{aligned}$$

Hence $\mathrm{D}U(g)=0$ $\varXi$-a.e. on $B_{(1-\varepsilon)r}(g_0)$. Since ε is arbitrary, we obtain $\mathrm{D}U=0$ $\varXi$-a.e. on $B_r(g_0)$.

Next the statement trivially follows from (19). The proposition is proved. □

We also give some variant of the local property of $(\mathcal{E},\mathbb{D})$ which will be needed in Section 7. The definition is taken from [5, 9].

Lemma 13 *For each* $U\in\mathbb{D}$ *and* $F,G\in C_b^1(\mathbb{R})$ *with* $\operatorname{supp}F\cap\operatorname{supp}G=\emptyset$,

$$\mathcal{E}(F(U)-F(0),G(U)-G(0))=0.$$

Proof The lemma immediately follows from Proposition 5. □

6.3 Strictly quasi-regularity and conservativeness in a partial case

In this subsection we will suppose that ξ is constant on some neighbourhoods of 0 and 1, i.e. there exists $\delta \in \left(0, \frac{1}{2}\right)$ such that $\xi(u) = \xi(0)$, $u \in [0, \delta)$, and $\xi(u) = \xi(1)$, $u \in (1-\delta, 1]$. Also, we set

$$h_1 = \frac{1}{\delta}\mathbb{I}_{[0,\delta)} \quad \text{and} \quad h_2 = \frac{1}{\delta}\mathbb{I}_{[1-\delta,1]}. \tag{23}$$

In this case, the space $L_2^{\uparrow}(\xi)$ is locally compact, that immediately follows from Lemma 5.1 [31] and the following lemma.

Lemma 14 *For all $p \geq 2$ and $g \in L_2^{\uparrow}(\xi)$ $\|g\|_p \leq |\langle g, h_1\rangle| \vee |\langle g, h_2\rangle| \leq \frac{1}{\sqrt{\delta}}\|g\|_2$.*

Proof Since $g \in L_2^{\uparrow}(\xi)$, Proposition 14 implies that g is constant on $[0, \delta)$ and $(1-\delta, 1]$. So,

$$\langle g, h_1\rangle = g(0) \quad \text{and} \quad \langle g, h_2\rangle = g(1).$$

Moreover, $|g(u)| \leq |g(0)| \vee |g(1)|$ for all $u \in (0, 1)$, since $g \in D^{\uparrow}$. Hence, using the Cauchy-Schwarz inequality, we obtain

$$\|g\|_p \leq |g(0)| \vee |g(1)| = |\langle g, h_1\rangle| \vee |\langle g, h_2\rangle| \leq \frac{1}{\sqrt{\delta}}\|g\|_2.$$

The lemma is proved. □

Proposition 10 *The Dirichlet form $(\mathcal{E}, \mathbb{D})$ is strictly quasi-regular and conservative.*

Proof To prove the strictly quasi-regularity, it is enough to check that $(\mathcal{E}, \mathbb{D})$ is regular[3] according to Proposition V.2.12 [39]. Hence, it is needed to prove that $\mathcal{FC}$ is dense in $C_0(L_2^{\uparrow}(\xi))$ with respect to the uniform norm, where $C_0(L_2^{\uparrow}(\xi))$ denotes the space of continuous functions on $L_2^{\uparrow}(\xi)$ with compact supports. But this easily follows from the Stone-Weierstrass theorem, Remark 11 and the fact that each ball in $L_2^{\uparrow}(\xi)$ is a compact set.

The conservativeness of $(\mathcal{E}, \mathbb{D})$ will follow from Theorem 1.6.6 [24]. Thus, it is enough to show that there exists a sequence $\{U_n,\ n \geq 1\} \subset \mathbb{D}$ such that

$$0 \leq U_n \leq 1, \quad \lim_{n\to\infty} U_n = 1 \quad \Xi\text{-a.e.} \tag{24}$$

and

$$\lim_{n\to\infty} \mathcal{E}(U_n, V) = 0 \tag{25}$$

for all $V \in \mathbb{D} \cap L_1(L_2^{\uparrow}(\xi), \Xi)$.

For each $n \in \mathbb{N}$ we take a function $\psi_n \in C_0^{\infty}(\mathbb{R})$ satisfying

- $\operatorname{supp}\psi_n \subset [-2n-1, 2n+1]$, $\psi(x) = 1$ on $[-n, n]$ and $\psi_n(x) \in [0, 1]$ for $n < |x| < 2n+1$;

[3] see e.g. the definition on p.118 [39]

– $|\psi_n'(x)| \leq \frac{1}{n}$ and $|\psi_n''(x)| \leq \frac{C}{n}$ for all $x \in \mathbb{R}$ and a constant C that is independent of n.

Also, we set

$$U_n(g) = u_n(\langle g, h_1\rangle, \langle g, h_2\rangle), \quad g \in L_2^{\uparrow}(\xi) \text{ and } n \geq 1,$$

where $u_n(x,y) = \psi_n(x)\psi_n(y)$, $x, y \in \mathbb{R}$, and h_1, h_2 are defined by (23). Then by Lemma 14, for each $\varphi \in C_0^\infty(\mathbb{R})$ satisfying $\varphi = 1$ on $[-(2n+1)^2, (2n+1)^2]$ the equality

$$U_n(g) = U_n(g)\varphi(\|g\|_2^2), \quad g \in L_2^{\uparrow}(\xi),$$

holds. This implies that $U_n \in \mathcal{FC}$ and

$$\begin{aligned} LU &= \frac{1}{2}\sum_{i,j=1}^{2} \partial_i\partial_j u_n(\langle g, h_1\rangle, \langle g, h_2\rangle)\langle \operatorname{pr}_g h_i, \operatorname{pr}_g h_j\rangle \\ &\quad + \frac{1}{2}\sum_{j=1}^{2} \partial_j u_n(\langle g, h_1\rangle, \langle g, h_2\rangle)\langle \xi - \operatorname{pr}_g \xi, h_j\rangle, \quad g \in L_2^{\uparrow}(\xi), \end{aligned}$$

for all $n \geq 1$, where L is defined by (18). By the construction of U_n, $\{U_n,\ n \geq 1\}$ satisfies (24). Moreover, using the Cauchy-Schwarz inequality, the trivial inequality $\|\operatorname{pr}_g h\|_2 \leq \|h\|_2$ and the dominated convergence theorem, we have for every $V \in \mathbb{D} \cap L_1(L_2^{\uparrow}(\xi), \Xi)$

$$\begin{aligned} \mathcal{E}(U_n, V) &= -(LU_n, V)_{L_2^{\uparrow}(\xi)} \\ &= \frac{1}{2}\sum_{i,j=1}^{2} \int_{L_2^{\uparrow}(\xi)} \partial_i\partial_j u_n(\langle g, h_1\rangle, \langle g, h_2\rangle)\langle \operatorname{pr}_g h_i, \operatorname{pr}_g h_j\rangle V(g)\Xi(dg) \\ &\quad + \frac{1}{2}\sum_{j=1}^{2} \int_{L_2^{\uparrow}(\xi)} \partial_j u_n(\langle g, h_1\rangle, \langle g, h_2\rangle)\langle \xi - \operatorname{pr}_g \xi, h_j\rangle V(g)\Xi(dg) \to 0 \end{aligned}$$

as $n \to \infty$. The proposition is proved. □

7 Intrinsic metric associated to $(\mathcal{E}, \mathbb{D})$

The aim of this section is to prove that L_2-metric is the intrinsic metric associated to $(\mathcal{E}, \mathbb{D})$ and to prove the analog of Varadhan's formula. For this we will use the result obtained in [5] (see also [27] for the Dirichlet forms on $L_2(\mu)$, where μ is a probability measure).

7.1 The boundedness of $\mathrm{D}U$ implies the Lipschitz continuity of U

In this section we prove that any function U from $\mathbb{D}$ with $\|\mathrm{D}U\| \leq 1$ Ξ-a.e. is 1-Lipschitz continuous.

Proposition 11 *Let $U \in \mathbb{D}$ and $\|\mathrm{D}U\|_2 \leq 1$ Ξ-a.e. on a convex open set $B \subseteq L_2^{\uparrow}(\xi)$. Then U has an 1-Lipschitz modification $\widetilde{U}$ on B, i.e. there exists a function $\widetilde{U} : B \to \mathbb{R}$ such that $\Xi\{g \in B : \ \widetilde{U}(g) \neq U(g)\} = 0$ and*

$$|\widetilde{U}(g_1) - \widetilde{U}(g_0)| \leq \|g_1 - g_0\|_2 \tag{26}$$

for all $g_0, g_1 \in B$.

Remark 16 If $U \in \mathcal{FC}$, then

$$U(g_1) - U(g_0) = \int_0^1 \langle \mathrm{D}U(g_t), g_1 - g_0 \rangle dt$$

for all $g_0, g_1 \in \mathcal{S}^{\uparrow}$, where $g_t = g_0 + t(g_1 - g_0)$. This follows from the trivial fact that $\sigma^\star(g_t) \supseteq \sigma^\star(g_1 - g_0)$ for all $t \in (0,1)$ and $g_0, g_1 \in \mathcal{S}^{\uparrow}$. Consequently, in this case the statement holds.

Proof (Proof of Proposition 11) Step I. First we show that for each $n \geq 1$, (26) holds Ξ_n-a.e on B. Let $n \geq 2$ be fixed. Since $\mathcal{FC}$ is dense in $\mathbb{D}$ (w.r.t. $\mathcal{E}^{\frac{1}{2}}$-norm), there exists a sequence $\{U_k\}_{k \geq 1} \subset \mathcal{FC}$ such that $U_k \to U$ and $\|\mathrm{D}U_k - \mathrm{D}U\|_2 \to 0$ in $L_2(L_2^{\uparrow}(\xi), \Xi)$ as $k \to \infty$. Hence, they converge in $L_2(L_2^{\uparrow}(\xi), \Xi_n)$.

Let $A \subseteq B$ such that $\Xi(B \setminus A) = 0$ and $\|\mathrm{D}U(g)\| \leq 1$ for all $g \in A$. We denote

$$A_n = A \cap \{\chi_n(q,x) : \ q \in Q^n, \ x \in E_0^n\}.$$

Then by Remark 4 and Lemma 2 (iii), $\Xi_n(B \setminus A_n) = 0$. Since Ξ_n is the push forward of the measure $\mu_\xi^n \otimes \lambda_n$ under the map χ_n (see Lemma 2 (i)), it is easy to see that there exists $Q_1 \subseteq Q^n$ such that $\mu_\xi^n(Q^n \setminus Q_1) = 0$ and $\lambda_n(B(q) \setminus A_n(q)) = 0$ for all $q \in Q_1$, where $A_n(q) = \{x \in E_0^n : \ \chi_n(q,x) \in A_n\}$ and $B(q) = \{x \in E_0^n : \ \chi_n(q,x) \in B\}$.

Next, we note that

$$\begin{aligned}
&\int_{L_2^{\uparrow}(\xi)} |U_k(g) - U(g)|^2 \Xi_n(dg) \\
&= \int_{Q^n} \left[\int_{E^n} |U_k(\chi_n(q,x)) - U(\chi_n(q,x))|^2 \lambda_n(dx) \right] \mu_\xi^n(dq) \to 0
\end{aligned}$$

and, similarly,

$$\int_{Q^n} \left[\int_{E^n} \|\mathrm{D}U_k(\chi_n(q,x)) - \mathrm{D}U(\chi_n(q,x))\|_2^2 \lambda_n(dx) \right] \mu_\xi^n(dq) \to 0$$

as $k \to \infty$. Consequently, we can choose a subsequence $\{k'\} \subseteq \mathbb{N}$ (we assume that $\{k'\}$ coincides with $\mathbb{N}$ without loss of generality) and a measurable subset $Q_2 \subseteq Q^n$ such that $\mu_\xi^n(Q^n \setminus Q_2) = 0$ and for all $q \in Q_2$

$$\int_{E^n} |U_k(\chi_n(q,x)) - U(\chi_n(q,x))|^2 \lambda_n(dx) \to 0, \\ \int_{E^n} \|\mathrm{D}U_k(\chi_n(q,x)) - \mathrm{D}U(\chi_n(q,x))\|_2^2 \lambda_n(dx) \to 0 \tag{27}$$

as $k \to \infty$.

Let $q \in Q_1 \cap Q_2$ be fixed and

$$f_k(x) := U_k(\chi_n(q,x)), \quad x \in E_0^n, \\ f(x) := U(\chi_n(q,x)), \quad x \in E_0^n.$$

Then f_k, $k \geq 1$, belong to $C_0^\infty(E^n)$ and

$$\mathrm{D}U_k(\chi_n(q,x)) = \sum_{i=1}^n \frac{\partial f_k(x)}{\partial x_i} \frac{\mathbb{I}_{[q_{i-1},q_i)}}{q_i - q_{i-1}}, \quad x \in E_0^n, \tag{28}$$

by Lemma 7. We are going to show that $\mathrm{D}U(\chi_n(q,\cdot))$ is also given by (28), where the partial derivatives of f_k is replaced by the Sobolev partial derivatives of f.

So, first we note that $\mathrm{D}U(\chi_n(q,\cdot))$ can be given as follows

$$\mathrm{D}U(\chi_n(q,x)) = \sum_{i=1}^n \widetilde{f}^i(x) \frac{\mathbb{I}_{[q_{i-1},q_i)}}{q_i - q_{i-1}}, \quad x \in E_0^n,$$

for some measurable functions $\widetilde{f}^i : E_0^n \to \mathbb{R}$, since the set $\left\{\sum_{i=1}^n x_i \mathbb{I}_{[q_{i-1},q_i)},\ x \in \mathbb{R}^n\right\}$ is closed in $L_2(\xi)$. Moreover, by (27), we have that

$$\int_{E_0^n} |f_k(x) - f(x)|^2 \lambda_n(dx) \to 0$$

and

$$\int_{E_0^n} \sum_{i=1}^n \left[\widetilde{f}^i(x) - \frac{\partial f_k(x)}{\partial x_i}\right]^2 (q_i - q_{i-1}) \lambda_n(dx) \to 0$$

as $k \to \infty$. It immediately implies that f belongs to the Sobolev space $H^{1,2}(E_0^n)$ with $\widetilde{f}^i = \frac{\partial f}{\partial x_i}$. In particular,

$$\int_{\mathbb{R}^n} f(x) \frac{\partial \varphi(x)}{\partial x_i} dx = -\int_{\mathbb{R}^n} \widetilde{f}^i(x) \varphi(x) dx. \tag{29}$$

for each $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \varphi \subset E_0^n$ and f, $\widetilde{f}^i$, $i \in [n]$, equal zero outside E^n.

Next, let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be a non negative function with

$$\int_{\mathbb{R}^n} \varphi(x) dx = 1.$$

Then the convolution

$$f_\varepsilon(x) = f * \varphi_\varepsilon(x) = \int_{\mathbb{R}^n} f(y)\varphi_\varepsilon(x-y)dy, \quad x \in \mathbb{R}^n,$$

where $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x\varepsilon^{-1})$, belongs to $C^\infty(\mathbb{R}^n)$ and converges to f λ_n-a.e. on E_0^n. Moreover, by (29),

$$\frac{\partial f_\varepsilon(x)}{\partial x_i} = \widetilde{f}^i * \varphi_\varepsilon(x)$$

for every $x \in E_0^n$ and all $\varepsilon > 0$ satisfying $\operatorname{supp}\varphi_\varepsilon(x-\cdot) \subset E_0^n$.

We recall that $B(q) = \{x \in E_0^n : \ \chi_n(q,x) \in B\}$. Let $B(q) \neq \emptyset$. It is easily seen that $B(q)$ is an open convex subset of E_0^n. Then for each $x \in B(q)$ and $\varepsilon > 0$ such that $\operatorname{supp}\varphi_\varepsilon(x-\cdot) \subset B(q)$ we can estimate

$$\begin{aligned}
\sum_{i=1}^n \left(\frac{\partial f_\varepsilon(x)}{\partial x_i}\right)^2 \frac{1}{q_i - q_{i-1}} &= \sum_{i=1}^n \left(\widetilde{f}^i * \varphi_\varepsilon(x)\right)^2 \frac{1}{q_i - q_{i-1}} \\
&\leq \sum_{i=1}^n \int_{\mathbb{R}^n} (\widetilde{f}^i(y))^2 \varphi_\varepsilon(x-y)dy \frac{1}{q_i - q_{i-1}} = \int_{\mathbb{R}^n} \sum_{i=1}^n \frac{(\widetilde{f}^i(y))^2}{q_i - q_{i-1}} \varphi_\varepsilon(x-y)dy \\
&= \int_{E_0^n} \|\mathrm{D}U(\chi_n(q,y))\|_2^2 \varphi_\varepsilon(x-y)\lambda_n(dy) \\
&= \int_{B(q)} \|\mathrm{D}U(\chi_n(q,y))\|_2^2 \varphi_\varepsilon(x-y)\lambda_n(dy) \leq 1,
\end{aligned} \tag{30}$$

since $\|\mathrm{D}U(\chi_n(q,\cdot))\|_2 \leq 1$ λ_n-a.e. on $B(q)$.

Let $x^0, x^1 \in B(q)$ and $\varepsilon_0 > 0$ such that $f_\varepsilon(x^i) \to f(x^i)$ and $\operatorname{supp}\varphi_{\varepsilon_0}(x^i - \cdot) \subset B(q)$, $i = 0,1$. Using the convexity of $B(q)$, it is easy to see that

$$\operatorname{supp}\varphi_{\varepsilon_0}(x^t - \cdot) \subset B(q), \quad t \in (0,1),$$

where $x^t = x^0 + t(x^1 - x^0)$. By Hölder's inequality and (30), we can estimate

$$\begin{aligned}
(f_\varepsilon(x^1) - f_\varepsilon(x^0))^2 = \left(\int_0^1 \frac{d}{dt} f_\varepsilon(x^t)dt\right)^2 &= \left(\int_0^1 \sum_{i=1}^n \partial_i f_\varepsilon(x^t)(x_i^1 - x_i^0)dt\right)^2 \\
&\leq \int_0^1 \sum_{i=1}^n \left(\partial_i f_\varepsilon(x^t)\right)^2 \frac{1}{q_i - q_{i-1}} dt \sum_{i=1}^n (x_i^1 - x_i^0)^2 (q_i - q_{i-1}) \\
&\leq \|\chi_n(q,x^1) - \chi_n(q,x^0)\|_2^2
\end{aligned}$$

for all $\varepsilon \in (0, \varepsilon_0]$. Hence using the convergence of $f_\varepsilon(x^i)$ to $f(x^i)$, $i = 0,1$, and the previous estimate, we have that

$$|U(\chi(q,x^1)) - U(\chi(q,x^0))| \leq \|\chi_n(q,x^1) - \chi_n(q,x^0)\|_2. \tag{31}$$

Since (31) holds for all $q \in Q_1 \cap Q_2$ and $x^i \in B(q)$, $i = 0,1$, such that $f_\varepsilon(x^i) \to f(x^i)$ as $\varepsilon \to 0$, we have that

$$|U(g_1) - U(g_0)| \leq \|g_1 - g_0\|_2, \quad \Xi_n\text{-a.e. on } \ B, \tag{32}$$

due to the equalities $\mu_\xi^n(Q^n \setminus (Q_1 \cap Q_2)) = 0$ and $\lambda_n\{x \in B(q) : f_\varepsilon(x) \not\to f(x)\} = 0$.

We also note that using the same argument, we can show that (31) holds $\varXi_1$-a.e. on B.

Step II. Let $\widetilde{B}_n \subseteq B \cap \operatorname{supp} \varXi_n$ such that $\varXi_n(B \setminus \widetilde{B}_n) = 0$ and for all $g_0, g_1 \in \widetilde{B}_n$ (32) holds. Since $\varXi_n(B \setminus \widetilde{B}_n) = 0$, $\widetilde{B}_n$ is dense in $B \cap \operatorname{supp} \varXi_n$. Consequently, there exists a unique 1-Lipschitz function $\widetilde{U}_n : B \cap \operatorname{supp} \varXi_n \to \mathbb{R}$ that is the extension of U to $B \cap \operatorname{supp} \varXi_n$. Moreover, $\widetilde{U}_n = U$ $\varXi_n$-a.e. By the uniqueness of the extension and Corollary 1, we have that $\widetilde{U}_n = \widetilde{U}_{n+1}$ on $B \cap \operatorname{supp} \varXi_n = B \cap \{g \in L_2^\uparrow :\ \sharp g \leq n\}$. So, we can well define

$$\widetilde{U}_\infty(g) = \widetilde{U}_n(g), \quad g \in B \cap \operatorname{supp} \varXi_n = B \cap \{g \in L_2^\uparrow :\ \sharp g \leq n\}.$$

Thus, $\widetilde{U}_\infty$ is an 1-Lipschitz function defined on $B \cap (\bigcup_{n=1} \operatorname{supp} \varXi_n) = B \cap \mathcal{S}^\uparrow$, since for any $g_0, g_1 \in B \cap \mathcal{S}^\uparrow$ there exists $n \in \mathbb{N}$ such that $g_0, g_1 \in B \cap \{g \in L_2^\uparrow :\ \sharp g \leq n\}$. By the density of $B \cap \mathcal{S}^\uparrow$ in B, we can extend $\widetilde{U}_\infty$ to an 1-Lipschitz function $\widetilde{U}$ defined on B, moreover $\widetilde{U} = U$ $\varXi$-a.e. on B because $\varXi(L_2^\uparrow(\xi) \setminus \mathcal{S}^\uparrow) = 0$ (see Corollary 2). It proves the proposition. □

7.2 Intrinsic metric and Varadhan's formula

Since the measure $\varXi$ is σ-finite, we will define the intrinsic metric associated to $(\mathcal{E}, \mathbb{D})$ using a localization of the domain $\mathbb{D}$ (see [5]). Let $L_0(\varXi)$ denote the set of all measurable functions on $L_2^\uparrow(\xi)$ and $K_n := \{g \in L_2^\uparrow(\xi) :\ \|g\|_2 \leq n\}$, $n \in \mathbb{N}$. Then the family of balls $\{K_n\}_{n \geq 1}$ satisfies the following conditions

$(N1)$ For every $n \in \mathbb{N}$ there exists $V_n \in \mathbb{D}$ such that $V_n \geq 1$ $\varXi$-a.e. on K_n;
$(N2)$ $\bigcup_{n=1}^\infty \mathbb{D}_{K_n}$ is dense in $\mathbb{D}$ (w.r.t. $\mathcal{E}^{\frac{1}{2}}$-norm).

Remark 17 We note that the family $\{K_n\}_{n\geq 1}$ is a nest. It is also a nest according the definition given in [5], where the topology (on $L_2^\uparrow(\xi)$) is not needed.

We set

$$\mathbb{D}_{loc}(\{K_n\}) = \left\{ U \in L_0(\varXi) :\ \begin{array}{l} \text{there exists } \ \{U_n\}_{n\geq 1} \subset \mathbb{D} \ \text{ such that} \\ U = U_n \ \ \varXi\text{-a.e. on } \ K_n \ \text{ for each } \ n \end{array} \right\}$$

and let $\mathbb{D}_{loc,b}(\{K_n\})$ denote the set of all essentially bounded functions from $\mathbb{D}_{loc}(\{K_n\})$. For $U, V \in \mathbb{D}_b$, where $\mathbb{D}_b$ is the set of all essentially bounded functions from $\mathbb{D}$, we define

$$I_U(V) = 2\mathcal{E}(UV, U) - \mathcal{E}(U^2, V).$$

By the locality of $(\mathcal{E}, \mathbb{D})$ (see Lemma 13), $I_U(V)$ and $\mathrm{D}U$ can be well-defined for all $U \in \mathbb{D}_{loc,b}(\{K_n\})$ and $V \in \bigcup_{n=1}^\infty \mathbb{D}_{K_n,b}$, where $\mathbb{D}_{K_n,b} = \mathbb{D}_{K_n} \cap \mathbb{D}_b$, setting $I_U(V) = I_{U_n}(V)$ and $\mathrm{D}U = \mathrm{D}U_n$ if $V \in \mathbb{D}_{K_n,b}$ and $U_n = U$ $\varXi$-a.e. on K_n.

We set

$$\mathbb{D}_0 = \left\{ U \in \mathbb{D}_{loc,b}(\{K_n\}) : \quad I_U(V) \le \|V\|_{L_1(\Xi)} \ \text{ for every } \ V \in \bigcup_{n=1}^{\infty} \mathbb{D}_{K_n,b} \right\}.$$

Remark 18 According to Proposition 3.9 [5], the set $\mathbb{D}_0$ does not depend on the family of increasing sets $\{K_n\}_{n\ge1}$ that satisfies $(N1)$, $(N2)$.

Lemma 15 *The set* $\mathbb{D}_0$ *coincides with* $\{U \in \mathbb{D}_{loc,b}(\{K_n\}) : \quad \|\mathrm{D}U\|_2 \le 1 \quad \Xi\text{-a.e.}\}$.

Proof The statement easily follows from the relation

$$I_U(V) = \int_{L_2^{\uparrow}(\xi)} \|\mathrm{D}U(g)\|_2^2 V(g) \Xi(dg),$$

the density of $\mathcal{FC}_{K_n} = \{U \in \mathcal{FC} : \ U = 0 \ \ \Xi\text{-a.e. on } \ L_2^{\uparrow}(\xi) \setminus K_n\}$ in $L_1(K_n, \Xi)$ (w.r.t. L_1-norm) and the duality between $L_1(K_n, \Xi)$ and $L_\infty(K_n, \Xi)$. □

We note that each $U \in \mathbb{D}_0$ (or from $\mathbb{D}_{loc,b}(\{K_n\})$ satisfying $\|\mathrm{D}U\|_2 \le 1$ Ξ-a.e.) always has a continuous modification, by Proposition 11. Further, considering such a function, we will take its continuous modification.

Theorem 4 *The intrinsic metric for the Dirichlet form* $(\mathcal{E}, \mathbb{D})$ *is the* L_2*-metric, that is, for all* $g_0, g_1 \in L_2^{\uparrow}(\xi)$

$$\begin{aligned}\|g_1 - g_0\|_2 &= \sup_{U \in \mathbb{D}_0} \{U(g_1) - U(g_0)\} \\ &= \sup \{U(g_1) - U(g_0) : \quad U \in \mathbb{D}_{loc,b}(\{K_n\}), \quad \|\mathrm{D}U\|_2 \le 1 \quad \Xi\text{-a.e.}\}.\end{aligned}$$

Proof The equality

$$\sup_{U \in \mathbb{D}_0} \{U(g_1) - U(g_0)\} = \sup \left\{ U(g_1) - U(g_0) : \quad \begin{matrix} U \in \mathbb{D}_{loc,b}(\{K_n\}), \\ |\mathrm{D}U\|_2 \le 1 \quad \Xi\text{-a.e.} \end{matrix} \right\}$$

follows from Lemma 15. Proposition 11 implies the lower bound

$$\|g_1 - g_0\| \ge \sup \left\{ U(g_1) - U(g_0) : \quad \begin{matrix} U \in \mathbb{D}_{loc,b}(\{K_n\}), \\ \|\mathrm{D}U\|_2 \le 1 \quad \Xi\text{-a.e.} \end{matrix} \right\}.$$

To finish the proof, for $g_0, g_1 \in L_2^{\uparrow}(\xi)$ and $g_0 \ne g_1$ we need to find $U \in \mathbb{D}_0$ such that $U(g_1) - U(g_0) = \|g_1 - g_0\|_2$. We take $u \in C_b^1(\mathbb{R})$ such that $u(x) = x$ for all $|x| \le \|g_1\|_2 \vee \|g_0\|_2$ and $|u'(x)| \le 1$, $x \in \mathbb{R}$, and define

$$U(g) = u\left(\frac{\langle g, g_1 - g_0 \rangle}{\|g_1 - g_0\|_2} \right), \quad g \in L_2^{\uparrow}(\xi).$$

Since $\frac{|\langle g_i, g_1 - g_0 \rangle|}{\|g_1 - g_0\|_2} \le \|g_0\|_2 \vee \|g_1\|_2$, we have

$$U(g_1) - U(g_0) = \|g_1 - g_0\|_2.$$

Moreover, it is easy to see that $U \in \mathbb{D}_{loc,b}$ and

$$\mathrm{D}U(g) = u'\left(\frac{\langle g, g_1 - g_0\rangle}{\|g_1 - g_0\|_2}\right)\frac{\mathrm{pr}_g(g_1 - g_0)}{\|g_1 - g_0\|_2},$$

by Proposition 5. Consequently, $\|\mathrm{D}U(g)\|_2 \le 1$ for all $g \in L_2^{\uparrow}(\xi)$. It proves the theorem. □

Next, let $\{T_t\}_{t\ge 0}$ denote the semigroup on $L_2(L_2^{\uparrow}(\xi), \Xi)$ associated with $(\mathcal{E}, \mathbb{D})$. For measurable sets $A, B \subseteq L_2^{\uparrow}(\xi)$ with positive Ξ-measure we define

$$P_t(A,B) = \int_{L_2^{\uparrow}(\xi)} \mathbb{I}_A(g)\cdot T_t\mathbb{I}_B(g)\Xi(dg)$$

and

$$\mathrm{d}(A,B) = \operatorname{ess\,inf}\{\|g - f\|_2 :\ g \in A,\ f \in B\}.$$

Theorem 5 *For any measurable $A, B \subset L_2^{\uparrow}(\xi)$ with $0 < \Xi(A) < \infty$, $0 < \Xi(B) < \infty$ and A or B open the relation*

$$\lim_{t\to 0} t \ln P_t(A,B) = -\frac{\mathrm{d}(A,B)^2}{2}$$

holds.

Proof The statement follows from the general result for symmetric diffusions obtained in [5] (see Theorem 2.7 there) and Theorem 4. □

The following result is a consequence of Theorem 5.2 [5] and Theorem 4. Let $\|g - A\|_2 := \operatorname{ess\,inf}_{f\in A}\|g - f\|_2,\ \ g \in L_2^{\uparrow}(\xi)$.

Theorem 6 *Let A be a non empty open subset of $L_2^{\uparrow}(\xi)$ with $\Xi(A) < \infty$ and Θ be any probability measure which is mutually absolutely continuous with respect to Ξ. Then the function $u_t = -t\ln T_t\mathbb{I}_A$ converges to $\frac{\|\cdot - A\|_2^2}{2}$ in the following senses.*

(a) $u_t \cdot \mathbb{I}_{\{u_t<\infty\}}$ converges to $\frac{\|\cdot - A\|_2^2}{2}\cdot \mathbb{I}_{\{\|\cdot - A\|_2<\infty\}}$ in Θ-probability as $t \to 0$.
(b) If F is a bounded function on $[0,\infty]$ that is continuous on $[0,\infty)$, then $F(u_t)$ converges to $F\left(\frac{\|\cdot - A\|_2^2}{2}\right)$ in $L_2(L_2^{\uparrow}(\xi), \Theta)$ as $t \to 0$.

8 Sticky-reflected particle system

In this section we study some properties of the process associated with the Dirichlet form $(\mathcal{E}, \mathbb{D})$. Let $X = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \{X_t\}_{t\ge 0}, \{\mathbb{P}_g\}_{g\in L_2^{\uparrow}(\xi)_\Delta}\right)$ be a Ξ-tight (Markov) diffusion[4] process with state space $L_2^{\uparrow}(\xi)$ and life time ζ that is properly associated with $(\mathcal{E}, \mathbb{D})$. Such a process X exists and is unique

[4] see Definition V.1.10 [39]

up to Ξ-equivalence according to theorems IV.6.4 and V.1.11 [39]. We recall that X is continuous on $[0,\zeta)$, i.e.

$$\mathbb{P}_g \{t \mapsto X_t \ \text{ is continuous on } \ [0,\zeta)\} = 1 \quad \text{for } \ \mathcal{E}\text{-q.e. } \ g \in L_2^{\uparrow}(\xi).$$

We also remark that by Proposition 10, $\mathbb{P}_g\{\zeta < \infty\} = 0$ for $\mathcal{E}$-q.e. $g \in L_2^{\uparrow}(\xi)$, if ξ is constant on some neighbourhoods of 0 and 1.

8.1 X as $L_2(\xi)$-valued semimartingale

In this section, we show that the process X_t, $t \in [0,\zeta)$, is a continuous local semimartingale in $L_2^{\uparrow}(\xi)$ under $\mathbb{P}_g$ for $\mathcal{E}$-q.e. $g \in L_2^{\uparrow}(\xi)$. Letting

$$M_t = X_t - \frac{1}{2}\int_0^t (\xi - \mathrm{pr}_{X_s}\xi)ds, \quad t \in [0,\zeta),$$

the following theorem holds.

Theorem 7 *There exists an $\mathcal{E}$-exceptional subset N of $L_2^{\uparrow}(\xi)$ such that for all $g \in L_2^{\uparrow}(\xi) \setminus N$ and each $(\mathcal{F}_t)$-stopping time τ satisfying $\mathbb{P}_g\{\tau < \zeta\} = 1$ and $\mathbb{E}_g\|X_t^\tau\|_2^2 < \infty$, $t \geq 0$, the process M_t^τ, $t \geq 0$, is a continuous square integrable $(\mathcal{F}_t)$-martingale under $\mathbb{P}_g$ in $L_2(\xi)$ with the quadratic variation*[5]

$$[M_\cdot^\tau]_t = \int_0^{t\wedge\tau} \mathrm{pr}_{X_s} ds, \quad t \geq 0,$$

where $X_t^\tau := X_{t\wedge\tau}$ and $M_t^\tau := M_{t\wedge\tau}$. In particular, for each $h_1, h_2 \in L_2(\xi)$ the processes $\langle M_t^\tau, h_i\rangle$, $t \geq 0$, $i \in [2]$, are continuous square integrable $(\mathcal{F}_t)$-martingales under $\mathbb{P}_g$ with the joint quadratic variation

$$[\langle M_\cdot^\tau, h_1\rangle, \langle M_\cdot^\tau, h_2\rangle]_t = \int_0^{t\wedge\tau} \langle \mathrm{pr}_{X_s} h_1, h_2\rangle ds, \quad t \geq 0.$$

Proof The statement easily follows from the martingale problem for X (see e.g. Theorem 3.4 (i) [3]) and the fact that for all $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi = 1$ on an interval $[-C, C]$ and $U(g) := \langle g, h\rangle \varphi(\|g\|_2^2)$, $g \in L_2^{\uparrow}(\xi)$, we have

$$\mathrm{D}U(g) = \mathrm{pr}_g h \quad \text{and} \quad LU(g) = \frac{1}{2}\langle \xi - \mathrm{pr}_g \xi, h\rangle$$

for all $g \in L_2^{\uparrow}(\xi)$ satisfying $\|g\|_2^2 \leq C$. □

Corollary 5 *If ξ is a constant on some neighbourhoods of 0 and 1, then for $\mathcal{E}$-q.e. $g \in L_2^{\uparrow}(\xi)$ $\mathbb{E}_g\|X_t\|_2^2 < \infty$, $t \geq 0$, and the process M_t, $t \geq 0$, is a continuous square integrable $(\mathcal{F}_t)$-martingale under $\mathbb{P}_g$ in $L_2(\xi)$ with the quadratic variation*

$$[M_\cdot]_t = \int_0^t \mathrm{pr}_{X_s} ds, \quad t \geq 0.$$

[5] see Definition 2.9 [25] for the precise definition of quadratic variation of Hilbert-space-valued martingales

Proof According to Theorem 7, we only have to show that the processes X_t, $t \geq 0$, and M_t, $t \geq 0$, are square integrable, i.e. $\mathbb{E}_g \|X_t\|_2^2 < \infty$ and $\mathbb{E}_g \|M_t\|_2^2 < \infty$, $t \geq 0$, for $\mathcal{E}$-q.e. $g \in L_2^{\uparrow}(\xi)$.

Let the $\mathcal{E}$-exceptional set N of $L_2^{\uparrow}(\xi)$ be as in the Theorem 7. We set

$$N' := N \cup \left\{ g \in L_2^{\uparrow}(\xi) : \ \mathbb{P}_g\{\zeta < \infty\} > 0 \right\}.$$

Then N' also is $\mathcal{E}$-exceptional, by Proposition 10.

Let h_1 and h_2 be defined by (23) and $g \in L_2^{\uparrow}(\xi) \setminus N'$. Then by Theorem 7, $\langle M_t, h_i \rangle$, $t \geq 0$, $i \in [2]$, are continuous local $(\mathcal{F}_t)$-martingales under $\mathbb{P}_g$ with the quadratic variations satisfying

$$[\langle M_\cdot, h_i \rangle]_t \leq t \|h_i\|_2^2, \quad t \geq 0.$$

Thus, by Fatou's lemma $\mathbb{E}_g \langle M_t, h_i \rangle^2 < \infty$ for all $t \geq 0$ and $i \in [2]$. Using Lemma 14 and the boundedness of ξ, it is easy to check that X and M are square integrable. This proves the corollary. □

8.2 Evolution of the empirical mass process

Let $\mathcal{P}_2$ denote the space of probability measures on $\mathbb{R}$ with the finite second moment. We recall that $\mathcal{P}_2$ is a Polish space with respect to the (quadratic) Wasserstein metric

$$d_{\mathcal{W}}(\nu_1, \nu_2) = \left(\inf_{\nu \in \chi(\nu_1, \nu_2)} \iint_{\mathbb{R}^2} |x - y|^2 \nu(dx, dy) \right)^{\frac{1}{2}},$$

where $\chi(\nu_1, \nu_2)$ denotes the set of all probability measures on $\mathbb{R}^2$ with marginals $\nu_1, \nu_2 \in \mathcal{P}_2$. We denote the push forward of the Lebesgue measure Leb on $[0, 1]$ under $g \in L_2^{\uparrow}(\xi)$ by ιg, that is,

$$\iota g(A) = \mathrm{Leb}\{u : \ g(u) \in A\}, \quad A \in \mathcal{B}(\mathbb{R}).$$

Remark 19 The map ι is bijective isometry between L_2 and $\mathcal{P}_2$ (see e.g. Section 2.1 [10]).

Let

$$\mu_t := \iota X(\cdot, t), \quad t \geq 0, \tag{33}$$

where $\iota \varDelta := \varDelta$. We are going to show that the process μ_t, $t \geq 0$, is a martingale solution on $[0, \zeta)$ of the stochastic partial differential equation

$$d\mu_t = \varGamma(\mu_t) dt + \mathrm{div}(\sqrt{\mu_t} dW_t), \tag{34}$$

with $\langle \alpha, \varGamma(\nu) \rangle = \frac{1}{2} \sum_{x \in \mathrm{supp}\, \nu} \alpha''(x)$, $\alpha \in C_0^\infty(\mathbb{R})$. In particular, it will imply that (34) has no unique solution, since the modified massive Arratia flow is a martingale solution of the same equation (see Section 1.3.1 [35]).

Proposition 12 *For each* $\alpha \in C_b^1(\mathbb{R})$ *and* $\varphi \in C_0^\infty(\mathbb{R})$ *the function*

$$U(g) = \int_0^1 \alpha(g(s))ds \cdot \varphi(\|g\|_2^2), \quad g \in L_2^\uparrow(\xi),$$

belongs to $\mathbb{D}$ *and*

$$\mathrm{D}U(g) = \alpha'(g)\varphi(\|g\|_2^2) + \int_0^1 \alpha(g(s))ds \cdot 2\varphi'(\|g\|_2^2)g, \quad g \in L_2^\uparrow(\xi).$$

Proof The proof is given in the appendix. □

Corollary 6 *Let* $\alpha_j \in C_b^1(\mathbb{R})$, $j \in [m]$, $\varphi \in C_0^\infty(\mathbb{R})$ *and* $u \in C_b^1(\mathbb{R}^m)$. *Then the function*

$$\begin{aligned} U(g) &= u\left(\int_0^1 \alpha_1(g(s))ds, \ldots, \int_0^1 \alpha_m(g(s))ds\right)\varphi(\|g\|_2^2) \\ &= u\left(\int_0^1 \boldsymbol{\alpha}(g(s))ds\right)\varphi(\|g\|_2^2), \quad g \in L_2^\uparrow(\xi), \end{aligned} \tag{35}$$

belongs to $\mathbb{D}$ *and*

$$\begin{aligned} \mathrm{D}U(g) = &\sum_{j=1}^m \partial_j u\left(\int_0^1 \boldsymbol{\alpha}(g(s))ds\right)\alpha'(g)\varphi(\|g\|_2^2) \\ &+ u\left(\int_0^1 \boldsymbol{\alpha}(g(s))ds\right)\cdot 2\varphi'(\|g\|_2^2)g, \quad g \in L_2^\uparrow(\xi). \end{aligned} \tag{36}$$

Proof We take

$$\begin{aligned} &F(x_1, \ldots, x_m, x_{m+1}) = u(x_1, \ldots, x_m)x_{m+1}, \quad x \in \mathbb{R}^{m+1}, \\ &V_j(g) = \int_0^1 \alpha_j(g(s))ds \cdot \varphi_j(\|g\|_2^2), \quad g \in L_2^\uparrow(\xi), \quad j \in [m], \\ &V_{m+1}(g) = \varphi(\|g\|_2^2), \quad g \in L_2^\uparrow(\xi), \end{aligned}$$

where $\varphi_j \in C_0^\infty(\mathbb{R})$ with $\varphi_j = 1$ on $\operatorname{supp}\varphi$. Then by propositions 5 and 12, the function

$$U(g) = u\left(\int_0^1 \boldsymbol{\alpha}(g(s))ds\right)\varphi(\|g\|_2^2) = F(V_1(g), \ldots, V_{m+1}(g)), \quad g \in L_2^\uparrow(\xi),$$

belongs to $\mathbb{D}$ and (36) holds. The corollary is proved. □

Proposition 13 *Let* $\alpha_j \in C_b^2(\mathbb{R})$, $j \in [m]$, $\varphi \in C_0^\infty(\mathbb{R})$, $u \in C_b^2(\mathbb{R}^m)$ *and the function* U *be given by* (35). *Then* U *belongs to the domain of the generator* L

of the Dirichlet form $\mathcal{E}$, that is Friedrich's extension of $(L, \mathcal{FC})$ (see (18) for the definition of L on $\mathcal{FC}$). Moreover,

$$\begin{aligned}
LU(g) = \frac{1}{2}\Bigg[\sum_{i,j=1}^{m} \partial_i\partial_j u\left(\int_0^1 \boldsymbol{\alpha}(g(s))ds\right) \cdot \int_0^1 \alpha_i'(g(s))\alpha_j'(g(s))ds \\
+ \sum_{j=1}^{m} \partial_j u\left(\int_0^1 \boldsymbol{\alpha}(g(s))ds\right) \cdot \int_0^1 \frac{\alpha_j''(g(s))}{m_g(s)}ds\Bigg] \varphi(\|g\|_2^2) \\
+ \sum_{j=1}^{m} \partial_j u\left(\int_0^1 \boldsymbol{\alpha}(g(s))ds\right) \varphi'(\|g\|_2^2) \int_0^1 \alpha'(g(s))g(s)ds \\
+ u\left(\int_0^1 \boldsymbol{\alpha}(g(s))ds\right)\left[2\varphi''(\|g\|_2^2)\|g\|_2^2 + \varphi'(\|g\|_2^2)\cdot \sharp g\right], \quad g \in \mathcal{S}^{\uparrow} \cap L_2^{\uparrow}(\xi),
\end{aligned} \tag{37}$$

where $m_g(s) = \mathrm{Leb}\{r \in [0,1] : \ g(r) = g(s)\} = \mathrm{Leb}\, g^{-1}(g(s)), \ \ s \in [0,1]$.

Proof To prove the proposition, it is enough to show that for each $V \in \mathcal{FC}$

$$\mathcal{E}(U, V) = -\langle LU, V\rangle_{L_2(\varXi)},$$

where LU is defined by (37). The proof of this fact is similar to the proof of Theorem 3, using the trivial relation $\mathrm{D}U = \mathrm{pr}_{.}\nabla^{L_2}U = \nabla^{L_2}U$. □

We set

$$M_\alpha'(t) := \langle \alpha, \mu_t\rangle - \langle \alpha, \mu_0\rangle - \int_0^t \varGamma(\mu_s)ds, \quad t \geq 0,$$

where $\langle \alpha, \varGamma(\nu)\rangle = \frac{1}{2}\sum_{x \in \mathrm{supp}\,\nu} \alpha''(x), \quad \alpha \in C_0^\infty(\mathbb{R})$. Using the martingale problem for X and Proposition 13, it is easy to obtain the following statement.

Theorem 8 *There exists an $\mathcal{E}$-exceptional subset N of $L_2^{\uparrow}(\xi)$ such that for all $g \in L_2^{\uparrow}(\xi) \setminus N$, $\alpha \in C_0^\infty(\mathbb{R})$ and each $(\mathcal{F}_t)$-stopping time τ satisfying $\mathbb{P}_g\{\tau < \zeta\} = 1$ and $\mathbb{E}_g d_{\mathcal{W}}(\mu_t^\tau, \mathrm{Leb})^2 < \infty$, $t \geq 0$, the process $M_\alpha^\tau(t)$, $t \geq 0$, is a continuous square integrable $(\mathcal{F}_t)$-martingale under $\mathbb{P}_g$ in $L_2(\xi)$ with the quadratic variation*

$$\int_0^{t\wedge\tau} \left\langle (\alpha')^2, \mu_s\right\rangle ds,$$

where μ_t, $t \geq 0$, is defined by (33), $\mu_t^\tau := \mu_{t\wedge\tau}$ and $M_\alpha^\tau(t) := M_\alpha(t \wedge \tau)$.

The theorem immediately implies that μ_t, $t \geq 0$, is a martingale solution of equation (34) on $[0, \tau]$.

Corollary 7 *If ξ is constant on some neighbourhoods of 0 and 1, then for $\mathcal{E}$-q.e. $g \in L_2^{\uparrow}(\xi)$ the process $M_\alpha(t)$, $t \geq 0$, is a continuous square integrable $(\mathcal{F}_t)$-martingale under $\mathbb{P}_g$ in $L_2(\xi)$ with the quadratic variation*

$$\int_0^t \left\langle (\alpha')^2, \mu_s \right\rangle ds,$$

where μ_t, $t \geq 0$, is defined by (33).

Proof The corollary follows from Theorem 8 and the fact that $\mathbb{E}_g \|X_t\|_2^2 < \infty$, $t \geq 0$, for $\mathcal{E}$-q.e. $g \in L_2^{\uparrow}(\xi)$ (see Corollary 5). $\square$

Letting for measurable sets $A, B \subset \mathcal{P}_2$ and $\nu \in \mathcal{P}_2$

$$\begin{aligned} d_{\mathcal{W}}(A, B) &= \operatorname{ess\,inf}\{d_{\mathcal{W}}(\nu_1, \nu_2) :\ \nu_1 \in A,\ \nu_2 \in B\}, \\ d_{\mathcal{W}}(\nu, A) &= \operatorname*{ess\,inf}_{\rho \in A} d_{\mathcal{W}}(\nu, \rho), \end{aligned}$$

we can prove the following theorem.

Theorem 9 *Let ξ be a strictly increasing function and Σ be the push forward of Ξ under the map ι. Then the following statements hold.*

(i) For any measurable $A, B \subset \mathcal{P}_2$ with $0 < \Sigma(A) < \infty$, $0 < \Sigma(B) < \infty$ and A or B open we have

$$\lim_{t \to 0} t \ln \int_A \mathbb{P}_{\iota^{-1}\nu}\{\mu_t \in B\} \Sigma(d\nu) = -\frac{d_{\mathcal{W}}(A, B)^2}{2}.$$

(ii) Let A be a non empty open subset of $\mathcal{P}_2$ with $\Sigma(A) < \infty$ and let Θ be any probability measure which is mutually absolutely continuous with respect to Σ. Then the function $v_t = -t \ln \mathbb{P}_{\iota^{-1}\cdot}\{\mu_t \in A\}$ converges to $\frac{d_{\mathcal{W}}(\cdot, A)^2}{2}$ in the following senses.

(a) $v_t \cdot \mathbb{I}_{\{v_t < \infty\}}$ converges to $\frac{d_{\mathcal{W}}(\cdot, A)^2}{2} \cdot \mathbb{I}_{\{d_{\mathcal{W}}(\cdot, A) < \infty\}}$ in Θ-probability as $t \to 0$.
(b) If F is a bounded function on $[0, \infty]$ that is continuous on $[0, \infty)$, then $F(v_t)$ converges to $F\left(\frac{d_{\mathcal{W}}(\cdot, A)^2}{2}\right)$ in $L_2(\mathcal{P}_2, \Theta)$ as $t \to 0$.

Proof The statement follows from theorems 5 and 6 and the isometry of $L_2^{\uparrow}(\xi) = L_2^{\uparrow}$ and $\mathcal{P}_2$. $\square$

A Appendix

A.1 $L_2^{\uparrow}(\xi)$-functions

Let ξ be a bounded function from $D^{\uparrow}$ and, as before, $L_2^{\uparrow}(\xi)$ denote the set of functions from $L_2^{\uparrow}$ that are $\sigma^\star(\xi)$-measurable.

Remark 20 (i) The space $L_2^{\uparrow}(\xi)$ is closed in $L_2^{\uparrow}$.

(ii) Let $f \in L_2^{\uparrow}(\xi)$ and g be its modification from $D^{\uparrow}$, then g is $\sigma^{\star}(\xi)$-measurable.

In this section we give a convenient description of each function $g \in L_2^{\uparrow}(\xi)$ using its right continuous modification.

Proposition 14 *A function $g \in L_2^{\uparrow}$ belongs to $L_2^{\uparrow}(\xi)$ if and only if for all $a < b$ from $[0,1]$ the equality $\xi(a) = \xi(b)$ implies $g(a) = g(b-)$ (Here, as usual, we take the modification of g that belongs to $D^{\uparrow}$).*

Proof Let $g \in L_2^{\uparrow}(\xi)$ and $\xi(a) = \xi(b)$ for some $a < b$ and f is $\sigma(\xi)$ measurable with $g = f$ a.e. Note that such a function f exists according to Lemma 1.25 [28]. First, we note that the sets

$$\pi_r = \xi^{-1}(\{r\}) = \{s \in [0,1] : \xi(s) = r\},$$

are the smallest in $\sigma(\xi)$, i.e. for any non empty set $A \in \sigma(\xi)$ satisfying $A \subseteq \pi_r$ we have $A = \pi_r$. Consequently, the set

$$B = \{s \in [0,1] :\ f(a) = f(s)\} \cap \pi_{\xi(a)}$$

coincides with $\pi_{\xi(a)}$ (B is non empty, since $a \in B$). Next we note that $[a,b] \subseteq \pi_{\xi(a)} = B$, since ξ is non decreasing and $\xi(a) = \xi(b)$. Consequently, $f(a) = f(s)$ for all $s \in [a,b]$. So, trivially, the equality $f = g$ a.e. yields $g(a) = g(a+) = g(b-)$.

To prove the sufficiency, we first show that a function f is $\sigma(\xi)$ measurable, if f is Borel measurable and

$$\xi(a) = \xi(b) \ \text{ implies } \ f(a) = f(b) \ \text{ for all } \ a, b \in [0,1]. \tag{38}$$

Let us define the function $\eta[\xi(0), \xi(1)] \to [0,1]$, that will play a role of the inverse function for ξ, as follows

$$\eta(r) = \min\{s \in [0,1] :\ \xi(s) \geq r\}, \quad r \in [\xi(0), \xi(1)].$$

Then it is easy to see that η satisfies the following properties

a) η is a non decreasing left-continuous function;
b) $\eta(\xi(s)) = \widetilde{s}$, where $\widetilde{s} = \min\{\pi_{\xi(s)}\}$.

Using these properties and setting $\phi(r) = f(\eta(r))$, $r \in [\xi(0), \xi(1)]$, we can easily see that ϕ is a Borel function and

$$\phi(\xi(s)) = f(\eta(\xi(s))) = f(\widetilde{s}) = f(s), \quad s \in [0,1].$$

Thus, f is $\sigma(\xi)$-measurable, as a compositions of Borel function with ξ.

Let for all $a < b$ the equality $\xi(a) = \xi(b)$ implies $g(a) = g(b-)$. We are going to find a function f that satisfies (38) and coincides with g a.e. Denote the set of all discontinuous points of g by D_g that is at most countable, since g is non decreasing. Next, for all $b \in D_g$ we note that b satisfies only one of the following properties

– $\xi(a) \neq \xi(b)$ for all $a \neq b$;
– there exists $a < b$ such that $\xi(a) = \xi(b)$ and, consequently, $g(a) = g(b-)$;
– there exists $c > b$ such that $\xi(b) = \xi(c)$ and, consequently, $g(b) = g(c-)$.

Indeed, if there exist both a and c such that $a < b < c$ and $\xi(a) = \xi(b) = \xi(c)$ then $g(a) = g(c-)$. But it contradicts the assumption that b is a discontinuous point of g.

We define

$$f(s) = \begin{cases} g(s), & \text{if } \ s \in [0,1] \setminus D_g, \\ g(s), & \text{if } \ s \in D_g \ \text{ and } \ \xi(a) = \xi(s) \ \text{ for some } \ a < s, \\ g(s-), & \text{if } \ s \in D_g \ \text{ and } \ \xi(s) = \xi(c) \ \text{ for some } \ c > s. \end{cases}$$

Then f is a well-defined non decreasing function and, consequently, Borel measurable. Moreover, it is easily seen that f satisfies (38). So, f is $\sigma(\xi)$-measurable. Since D_g is at most countable and $\{s :\ g(s) \neq f(s)\} \subseteq D_g$, we have that $f = g$ a.e. So, g is $\sigma^{\star}(\xi)$-measurable, by Lemma 1.25 [28]. It finishes the proof. $\square$

A.2 Multivariate Bernstein polynomials

In this section we give a slight modification of the result stated in [58] about uniform approximation of a function and its partial derivatives by Bernstein polynomials.

For a function $f : [0,1]^k \to \mathbb{R}$ we define the Bernstein polynomials on $[0,1]^k$ as follows

$$\begin{aligned} B_n(f;x) = &\sum_{j_1,\dots,j_k=0}^{n} f\left(\frac{j_1}{n},\dots,\frac{j_k}{n}\right) C_n^{j_1}\dots C_n^{j_k} \\ &\cdot x_1^{j_1}(1-x_1)^{n-j_1}\dots x_k^{j_k}(1-x_k)^{n-j_k}, \end{aligned}$$

where $C_n^j = \frac{n!}{j!(n-j)!}$, $j \in [n] \cup \{0\}$.

Proposition 15 *If $f \in C^1(\mathbb{R}^k)$, then*

(i) $\{B_n(f;\cdot)\}_{n\geq 1}$ *uniformly converges to f on $[0,1]^k$;*
(ii) $\{\partial_i B_n(f;\cdot)\}_{n\geq 1}$ *uniformly converges to $\partial_i f$ on $[0,1]^k$ for all $i \in [k]$.*

Proof The statement is a partial case of Theorem 4 [58]. □

Next we would like to have a sequence of polynomials that approximate a function f on $[-M,M]^k$. We set for a fixed $M > 0$

$$\begin{aligned} f_M(x) &= f(2Mx - M), \\ P_n^M(f;x) &= B_n\left(f_M; \frac{x}{2M} + \frac{1}{2}\right) - B_n\left(f_M; \frac{1}{2}\right). \end{aligned} \tag{39}$$

We note that $P_n^M(f;0) = 0$. This property is important for us, since in this case the composition $P_n^M(f;U)$ belongs to $\mathcal{FC}$ for $U_i \in \mathcal{FC}$, $i \in [k]$, ($\mathcal{FC}$ is an associative algebra that does not contain constant functions).

The following proposition is a trivial consequence of the previous proposition.

Lemma 16 *Let $f \in C^1(\mathbb{R}^k)$ and $f(0) = 0$. Then $P_n^M(f;0) = 0$ and*

(i) $\{P_n^M(f;\cdot)\}_{n\geq 1}$ *uniformly converges to f on $[-M,M]^k$;*
(ii) $\{\partial_i P_n^M(f;\cdot)\}_{n\geq 1}$ *uniformly converges to $\partial_i f$ on $[-M,M]^k$ for all $i \in [k]$.*

A.3 Proof of auxiliary statements

A.3.1 Proof of Lemma 6

By Remark 5 (iii), $\mathrm{pr}_g\, h$ belongs to $L_2^{\uparrow}$. So, we need only to show that it has a modification taking a finite number of values. Consequently, using the linearity of pr_g and Remark 4, it is enough to prove that for any $H := [a,b) \subset [0,1]$, $\mathrm{pr}_g\, \mathbb{I}_H$ has a modification that takes at most three values.

We set $D_n = \left\{\frac{k}{2^n},\ k \in \mathbb{Z}\right\}$, $\mathcal{S}_n = \sigma\{[a,b) :\ a < b,\ a,b \in D_n\}$ and $\mathcal{F}_n = g^{-1}(\mathcal{S}_n)$. Let us note that $\{\mathcal{F}_n,\ n \in \mathbb{N}\}$, is increasing, since $\{\mathcal{S}_n,\ n \in \mathbb{N}\}$, is. Moreover, it is clear that

$$\sigma(g) = \bigvee_{n=1}^{\infty} \mathcal{F}_n = \sigma\left(\bigcup_{n=1}^{\infty} \mathcal{F}_n\right).$$

By Levi's theorem (see, e.g. Theorem 1.5 [38]),

$$\mathbb{E}(\mathbb{I}_H|\mathcal{F}_n) \to \mathbb{E}\left(\mathbb{I}_H \,\middle|\, \bigvee_{n=1}^{\infty} \mathcal{F}_n\right) \quad \text{a.e., as } n \to \infty, \tag{40}$$

where $\mathbb{E}$ denotes the expectation on the probability space $([0,1],\mathcal{B}([0,1]),\mathrm{Leb})$. Since each element of $\mathcal{F}_n$ can be written as a finite or a countable union of disjoint sets $G_{k,n} = g^{-1}\left(\left[\frac{k}{2^n},\frac{k+1}{2^n}\right)\right)$, $k\in\mathbb{Z}$, we obtain

$$\mathbb{E}(\mathbb{I}_H|\mathcal{F}_n) = \sum_{k\in\mathbb{Z}} \frac{\mathbb{I}_{G_{k,n}}}{\mathrm{Leb}(G_{k,n})}\mathbb{E}\mathbb{I}_{H\cap G_{k,n}}.$$

Next, by monotonicity of g, the set H can be covered by a finite number of $G_{k,n}$, i.e there exist integer numbers $p_1<p_2$ such that

- $\widetilde{H} := \bigcup_{k=p_1+1}^{p_2-1} G_{k,n} \subseteq H = [a,b)$;
- $a\in G_{p_1,n}$, $b\in G_{p_2,n}$;
- for each $k<p_1$ or $k>p_2$, $G_{k,n}\cap H=\emptyset$.

Thus,

$$\mathbb{E}(\mathbb{I}_H|\mathcal{F}_n) = \frac{\mathbb{I}_{G_{p_1,n}}}{\mathrm{Leb}(G_{p_1,n})}\mathbb{E}\mathbb{I}_{H\cap G_{p_1,n}} + \frac{\mathbb{I}_{G_{p_2,n}}}{\mathrm{Leb}(G_{p_2,n})}\mathbb{E}\mathbb{I}_{H\cap G_{p_2,n}} + \frac{\mathbb{I}_{\widetilde{H}}}{\mathrm{Leb}(\widetilde{H})}\mathbb{E}\mathbb{I}_{\widetilde{H}}.$$

Hence $\mathbb{E}(\mathbb{I}_H|\mathcal{F}_n)$ takes at most three values. By (40) and Remark 5, $\mathrm{pr}_g\,\mathbb{I}_H$ also takes at most three values. It proves the lemma.

A.3.2 Proof of Proposition 12

Here we will use the probabilistic approach. We will consider functions from $L_2^{\uparrow}(\xi)$ as random elements on the probability space $([0,1],\mathcal{B}([0,1]),\mathrm{Leb})$.

We note that the sequence of σ-algebras

$$\mathcal{S}_n = \sigma\left(\pi_i^n := \left[\frac{i-1}{2^n},\frac{i}{2^n}\right),\ i\in[2^n]\right),\quad n\in\mathbb{N},$$

increases to $\mathcal{B}([0,1])$. Thus, by the Levy theorem (see, e.g. Theorem 1.5 [38]), for each $g\in L_2^{\uparrow}(\xi)$

$$g_n := \mathbb{E}(g|\mathcal{S}_n) = \sum_{i=1}^{2^n}\langle g,h_i^n\rangle \mathbb{I}_{\pi_i^n} \to g \quad \text{a.s. as } n\to\infty,$$

where $h_i^n = 2^n\mathbb{I}_{\pi_i^n}$. Consequently, by the dominated convergence theorem,

$$\int_0^1 \alpha(g_n(s))ds = \sum_{i=1}^{2^n}\alpha(\langle g,h_i^n\rangle)\frac{1}{2^n} \to \int_0^1\alpha(g(s))ds \quad \text{as } n\to\infty.$$

Next we define

$$U_n(g) = \int_0^1\alpha(g_n(s))ds\cdot\varphi(\|g\|_2^2),\quad g\in L_2^{\uparrow}(\xi),$$

and note that $U_n\in\mathcal{FC}$. Moreover, for all $g\in L_2^{\uparrow}(\xi)$

$$\begin{aligned}\mathrm{D}U_n(g) &= \frac{1}{2^n}\sum_{i=1}^{2^n}\alpha'(\langle g,h_i^n\rangle)\,\mathrm{pr}_g\,h_i^n\varphi(\|g\|_2^2) + 2\int_0^1\alpha(g_n(s))ds\cdot\varphi'(\|g\|_2^2)g\\ &= \mathrm{pr}_g\,\alpha'(g_n)\varphi(\|g\|_2^2) + 2\int_0^1\alpha(g_n(s))ds\cdot\varphi'(\|g\|_2^2)g.\end{aligned}$$

By the dominated convergence theorem (for conditional expectations) and Remark 5 (ii),

$$\mathrm{pr}_g\,\alpha'(g_n) = \mathbb{E}(\alpha'(g_n)|\sigma^\star(g)) \to \mathbb{E}(\alpha'(g)|\sigma^\star(g)) = \alpha'(g) \quad \text{a.s. as } n\to\infty.$$

Thus, using the dominated convergence theorem again, we have

$$U_n \to U \quad \text{and} \quad \|\mathrm{D}U_n - \mathrm{D}U\|_2 \to 0 \quad \text{in} \quad L_2(\Xi) \quad \text{as} \quad n \to \infty,$$

where $U(g) = \int_0^1 \alpha(g(s))ds \cdot \varphi(\|g\|_2^2)$ and $\mathrm{D}U(g) = \alpha'(g)\varphi(\|g\|_2^2) + 2\int_0^1 \alpha(g(s))ds \cdot \varphi'(\|g\|_2^2)g$, $g \in L_2^{\uparrow}(\xi)$. The proposition is proved.

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